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Kharazmi University

10th Iranian Group Theory Conference

10th Iranian Group Theory Conference

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Plenary Talks

The classification of p-groups

Bettina Eick
Technische Universität Braunschweig, Germany

Abstract

In the talk we discuss different approaches to classify and investigate p-groups either theoretically or computationally using the computer algebra system GAP. We also exhibit how the available classifications are available in GAP and how these can be used.

Artin's conjecture

Sophie Morel
Princeton University, USA

Abstract

Artin's conjecture is a conjecture about finite-dimensional representations of Galois groups of finite extensions of number fields. It is one of the big open problems in number theory. I will explain the statement of the conjecture, and some of the tools in number theory and representation theory that are used to prove some cases.

A history of the order of the multipliers of p-groups and Lie algebras

Mohsen Parvizi
Ferdowsi University of Mashhad, Iran

Abstract

In this talk a brief history of the orders of the multipliers of p-groups and Lie algebras will be reviewed.

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Talks

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A class of finite groups with zero deficiency

Hossein Abdolzadeh

Abstract

We determine a new infinite sequence of finite groups with deficiency zero. The groups have 2 generators and 2 relations.

Keywords and phrases: deficiency zero, metacyclic group, finite group.

2010 *Mathematics subject classification:* Primary: 20F05; Secondary: 20D99.

1. Introduction

A group may has many presentations. A presentation $\langle X|R\rangle$ for a group G is said to be finite if X and R are both finite sets. A group is called finitely presented, if it has a finite presentation. For a finite presentation $\langle X|R\rangle$ of a group G the value $|X| - |R|$ is said to be the deficiency of the given presentation. The deficiency of G, denoted by $def(G)$, is defined to be the maximum of the deficiencies of all the finite presentations for G. We refer to [2] for a general introduction to the theory of presentations and to [1] for background on groups of deficiency zero.

Clearly if G is a finite group, then $def(G) \leq 0$. Each finite group of deficiency zero has trivial Schur multiplicator. However, looking for a deficiency zero presentation of a group which has trivial Schur multiplicator, is a long-standing question and many attempts have been made during the years on finite groups. All finite cyclic groups have deficiency zero. There are quite a number of examples of 2-generator groups of deficiency zero. It is known precisely which metacyclic groups have deficiency zero [6]. There are some nonmetacyclic groups with deficiency zero, see for example [3], [4] and [5]. A group G is called metacyclic if it has a cyclic normal subgroup N such that the quotient group G/N is also cyclic. In this paper we exhibit an infinite family of finite groups with zero deficiency which does not appear to be contained in the known classes. More precisely we prove the following theorem.

Theorem 1.1. *Let $n \neq 20$ be an integer and let $G(n)$ be the group defined by the presentation,*

$$G(n) = \langle x, y | x^n = y^4 = x^2(xy)^3 \rangle.$$

Then $G(n)$ is a finite group and the order of $G(n)$ is $24|20 - n|$ for n even and is $| 20 - n |$ for n odd. Moreover this family of groups contains infinitely many non-metacyclic groups.

We note that for $n = 20$, the group $G(20)$ is infinite[7].

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On connectivity of the prime index graph

Milad Ahanjideh* and Ali Iranmanesh

Abstract

Let G be a group. The prime index graph of G, denoted by $\Pi(G)$, is an undirected graph whose vertices are all subgroups of G and two distinct comparable subgroups H and K are adjacent if and only if $[H : K]$ or $[K : H]$ is prime. In this talk, it is shown that if G is a finite group and N is a normal solvable subgroup of G which its order is square free and $\Pi(G/N)$ is connected, then $\Pi(G)$ is connected.

Keywords and phrases: Prime index graph; Connectivity of graph .

2010 *Mathematics subject classification:* Primary: 05C25, 05C40; Secondary: 20D05, 20D06.

1. Introduction

One of the interesting topics in the last decade is the study of the structure of the groups by considering the properties of graphs associated to them or vice versa. On this matter, there are several graphs associated to groups, for instance the prime graph, the Cayley graphs and the power graph. Recently, in [2] a new graph which called the prime index graph has been introduced as follows:

Definition 1.1. [2] *Let G be a group. The prime index graph of G, denoted by $\Pi(G)$, is an undirected graph whose vertices are all subgroups of G and two distinct comparable subgroups H and K are adjacent if and only if $[H : K]$ or $[K : H]$ is prime.*

In Figure 1, the prime index graph of S_3 is given.

In [3], Bou-Rabee and Studemund, by considering the fixed prime p, define the other graph

* speaker

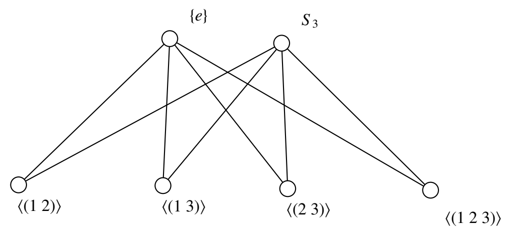

FIGURE 1. the prime index graph of S_3

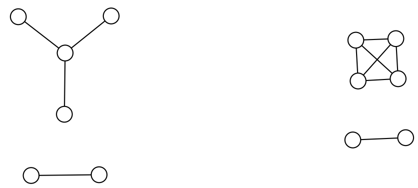

FIGURE 2. $\Gamma_2(S_3)$ and $\Gamma_3(S_3)$

associate to the groups, as follows:

Definition 1.2. [3] *Let p be a prime number and G be a group. The p-local commensurability graph, denoted by $\Gamma_p(G)$, is an undirected graph whose vertices are all subgroups of G and two distinct subgroups H and K are adjacent if and only if $[H : K \cap H][K : K \cap H]$ is a power of p.*

In Figure 2, $\Gamma_2(S_3)$ and $\Gamma_3(S_3)$ are given.

The connectivity of graphs associated to the groups is an example of the properties that can explain the structure of the groups. For instance, in [5], the structure of finite groups with disconnected prime graph has been determined. Pourgholi et al. [4] proved that if G is a nilpotent group which is not of prime power order, then the power graph of G is 2-connected.

In [2], it has been proved that every prime index graph on a solvable group is connected and if G is an infinite group, then $\Pi(G)$ is not connected. Moreover, they showed that the prime index graph of S_n is connected if and only if $n \leq 5$. In [1], the authors have improved the previous results by studying the connectivity of the prime index graph on non-solvable groups. In this talk, we continue investigating the prime index graphs and answer to a problem posed in [2] in a special case.

2. Main Results

Now, we state some lemmas without proof and use them in our proof.

Lemma 2.1. [2] *Let G be a finite group and N be a normal subgroup of G. If $\Pi(N)$ is a connected graph and also for every subgroup H/N of G/N, $\Pi(H/N)$ is a connected graph, then $\Pi(G)$ is connected.*

Lemma 2.2. [2] *Let G be a group and N be a normal subgroup of G. If $\Pi(G)$ is a connected graph, then $\Pi(N)$ and $\Pi(G/N)$ are connected graphs.*

Lemma 2.3. [2] *Let G be a finite solvable group. Then $\Pi(G)$ is connected.*

Lemma 2.4. [2] *Let G be a finite group and N be a normal subgroup of G. If $\Pi(N)$ is a connected graph and G/N is a solvable group, then $\Pi(G)$ is connected.*

Lemma 2.5. [2] *Let $G \cong H \times K$, for some groups H and K. If $\Pi(G)$ is connected, then both $\Pi(H)$ and $\Pi(K)$ are connected.*

On the matter of the above results, in [2], the authors posed the following problem: **Problem.** Let G be a group and N be a normal subgroup of G. If $\Pi(N)$ and $\Pi(G/N)$ are both connected, then is it true that $\Pi(G)$ is connected?

In the following, we answer to the above problem in a special case.

Theorem 2.6. *Let G be a finite group. If N is a normal solvable subgroup of G which its order is square free and $\Pi(G/N)$ is connected, then $\Pi(G)$ is connected.*

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Isoclinism in Moufang loops and characterization of finite simple Moufang loops by non-commuting graph and isoclinism

Karim Ahmadidelir

Abstract

The non-commuting graph of a non-abelian finite group has received some attention in existing literature. The order of groups in some classes of finite groups have been characterized by their non-commuting graphs. However, it has been proved recently that a finite simple group can be characterized by its non-commuting graph. We have already generalized the two notions, commutativity degree and non-commuting graph for a finite Moufang loop M and tried to characterize some finite non-commutative Moufang loops with their non-commuting graph. Also, it has been proved by Lescot that two isoclinic finite groups have the same commutativity degrees. In this talk, we want to generalize the notion of isoclinism to Moufang loops and show that two isoclinic finite Moufang loops have the same commutativity degrees. Then, we show that the finite Moufang loops with the same commutativity degrees and isomprphic non-commuting graphs are order characterictic and finally, chracterize all finite simple Moufang loops under isomorphism of their non-commuting graphs and isoclinism.

Keywords and phrases: Loop theory, Finite Moufang loops, Non-commuting graph in finite groups, Commutativity degree, Isoclinism. .

2010 *Mathematics subject classification:* Primary: 20N05, 20P05; Secondary: 20D05, 20B05.

1. Introduction

A set Q with one binary operation is a quasigroup if the equation $xy = z$ has a unique solution in Q whenever two of the three elements $x, y, z \in Q$ are specified. Loop is a quasigroup with a neutral element 1 satisfying $1x = x1 = x$ for every x. Moufang loops are loops in which any of the (equivalent) Moufang identities $((xy)x)z = x(y(xz))$, $x(y(zy)) = ((xy)z)y$, $(xy)(zx) = x((yz)x)$, $(xy)(zx) = (x(yz))x$ holds.

For a loop L, the *commutator* of x and y, and the *associator* of x, y and z in L, denoted by $[x, y]$ and $[x, y, z]$, are defined by $xy = (yx) \cdot [x, y]$ and $(xy)z = x(yz) \cdot [x, y, z]$, respectively. Commutant (or *Moufang center* or *centrum*) of Q is defined by $\{x \in Q \mid xy = yx, \forall y \in Q\}$ and is denoted by $C(Q)$. *Center* of Q is defined by $\{x \in Q \mid [x, y] = [x, y, z] = [y, x, z] = 1\}$ and is denoted by $Z(Q)$. *Nucleus* of Q is denoted by $N(Q)$ and is the subset $\{x \in Q \mid x(yz) = (xy)z,\ \ y(xz) = (yx)z,\ \ y(zx) = (yz)x,\ \ \forall y, z \in Q\}$. A non-empty subset P of Q is called a *subloop* of Q if P is itself a loop under the binary operation of Q; in particular if this operation is associative on P, then it is called a *subgroup* of Q. A subloop $N \leq Q$ is called *normal* if $xN = Nx$; $x(yN) = (xy)N$; $N(xy) = (Nx)y$ for every $x, y \in Q$. The *commutator-associator subloop* (also called the *derived subloop*) of a loop L, denoted by L', is the least normal subloop of L such that $\frac{L}{L'}$ is an abelian group. Hence L' is the least normal subloop of L containing all commutators $[x, y]$ and all associators $[x, y, z]$.

Now, $Z(Q) = C(Q) \cap N(Q)$, and $N(Q)$ and $Z(Q)$ are subgroups of Q, but in general, $C(Q)$ is not even a subloop. Of course, if Q be Moufang, then $C(Q)$ is a subloop of that (in fact, all of them, i.e. $N(Q)$, $Z(Q)$ and $C(Q)$, are normal in Q).

The *non-commuting graph* associated to a non-abelian group G, Γ_G, is a graph with vertex set $G \setminus Z(G)$ where distinct non-central elements x and y of G are joined by an edge if and only if $xy \neq yx$. Recently, many authors have studied the non-commuting graph associated to a non-abelian group. The order of groups in some classes of finite groups have been characterized by their non-commuting graphs (specially all finite simple groups and non-abelian nilpotent groups with irregular isomorphic non-commuting graphs), although the order of an arbitrary finite group can not be characterized by its non-commuting graph. Also, although, in general, a finite group can not be characterized by its non-commuting graph, however it has been proved recently that a finite simple group can be characterized by its non-commuting graph. For more details, see [1, 5, 7].

The author, have been generalized the notion of non-commuting graph to the finite loops and then characterized some small Moufang loops by their non-commuting graphs [2]. For a loop L, he has defined $L \setminus C(L)$, as the vertex set of the non-commuting graph of L, with two vertices x and y joined by an edge whenever the commutator of x and y is not the identity. The non-commuting graph of a finite non-abelian group is always connected with diameter two and girth 3. The same is true for the non-commuting graph of a finite non-commutative Moufang loop except that he has been able only to prove that the diameter is at most 6. Then he has tried to characterise some finite non-commutative Moufang loops with their non-commuting graph. Also, he obtained some results related to the non-commuting graph of a finite non-abelian Moufang loop. Finally, he has given a conjecture stating that the above result (i.e. the characterization of finite simple groups by their non-commuting graphs) is true for all finite simple Moufang loops. In the sequel, we will prove this conjecture with one additional hypothesis.

The *commutativity degree*, $Pr(G)$, of a finite group G (i.e. the probability that two -randomly chosen- elements of G commute with respect to its operation) has been studied well by many authors. In general, for a finite algebraic structure A, with at least one binary operation like as "$\cdot$", the commutativity degree of A is:

$$Pr(A) = \frac{|\{(x, y) \in A^2 \mid x \cdot y = y \cdot x\}|}{|A^2|}.$$

For a finite group A, it is proved that $Pr(A) = \frac{k(A)}{|A|}$, where $k(A)$ is the number of conjugacy classes of A. It is well-known that the best upper bound for $Pr(G)$ is $\frac{5}{8}$ for a finite non-abelian group G. Also, the author of this paper and his colleagues have shown in [4] that the $\frac{5}{8}$ is not an upper bound for $Pr(A)$, where A is a finite non-abelian semigroup and/or monoid.

The speaker of this talk has defined the same concept for a finite non-commutative *Moufang loop M* and tried to give a best upper bound for $Pr(M)$. He has proved that for a well-known class of finite Moufang loops, named *Chein loops*, and its modifications, this best upper bound is $\frac{23}{32}$ and conjectured that *for any finite Moufang loop M, $Pr(M) \leq \frac{23}{32}$*. Also, he has obtained some results related to the $Pr(M)$ and asked the similar questions raised and answered in group theory about the relations between the structure of a finite group and its commutativity degree in finite Moufang loops (for more details see [2]).

In this talk, we verify the relationship between the commutativity degree and the non-commuting graph of a finite Moufang loop. Then we generalize the notion of isoclinism to Moufang loops and finally, by use of these concepts and tools, characterize finite simple Moufang loops.

2. Main Results

In [6], Lescot showed that two isoclinic finite groups have the same commutativity degrees and then he classified all finite groups with commutativity degree greater than or equal to $\frac{1}{2}$ upto isoclinism. Here is a generalization of this concept to Moufang loops:

Definition 2.1. *Two Moufang loops M and L are called isoclinic if there are isomorphisms $\varphi : \frac{M}{Z(M)} \to \frac{L}{Z(L)}$ and $\psi : M' \to L'$ such that the following diagram commutes:*

$$\begin{array}{ccc} \frac{M}{Z(M)} \times \frac{M}{Z(M)} & \longrightarrow^{(\varphi,\varphi)} & \frac{L}{Z(L)} \times \frac{L}{Z(L)} \\ \downarrow^{\alpha} & \circlearrowright & \downarrow^{\beta} \\ M' & \longrightarrow^{\psi} & L' \end{array}$$

i.e., $\psi\alpha = \beta(\varphi, \varphi)$.

As in finite groups, we prove that the isoclinic finite Moufang loops have the same commutativity degrees. Then, we determine the structure of a Moufang loop which is isoclinic with a given non-commutative simple Moufang loop.

Theorem 2.2. *Let M and L be two finite isoclinic Moufang loops. Then $Pr(M) = Pr(L)$.*

Theorem 2.3. *Let S be a non-commutative simple Moufang loop. Then any Moufang loop M isoclinic to S is isomorphic to $S \times A$, for some commutative Moufang loop A.*

Theorem 2.4. *Let M be a Moufang loop and L be a subloop of M such that $M = LZ(M)$. Then M and L are isoclinic. The converse is true if L is finite.*

The following lemma gives an important relation between commutativity degree of a finite Moufang loop M and its non-commuting graph. It shows that if we know the sizes of M and edge set of Γ_M, then we can obtain $Pr(M)$, the commutativity degree of M, and vice versa, if we have the size of M and $Pr(M)$ then we will get the size of edge set of Γ_M.

Lemma 2.5. *Let M be a finite Moufang loop. Then $Pr(M) = \frac{|M|^2-2e}{|M|^2}$, where $Pr(M)$ is the commutativity degree of M and e is equal to the number of edges of its non-commuting graph.*

Here, we show that the finite Moufang loops with the same commutativity degrees and isomprphic non-commuting graphs are order characterictic.

Theorem 2.6. *Let M and L be two finite Moufang loops such that$Pr(M) = Pr(L)$ and also, $\Gamma_M \cong \Gamma_L$ (their non-commuting graphs are isomorphic). Then $|M| = |N|$. If M is centerless then L is too.*

By above results, we conclude that finite non-commutative simple Moufang loops can be characterized by isomorphsim of non-commuting graphs and isoclinism.

Corollary 2.7. *Let S be a finite simple Moufang loop and M be a Moufang loop such that $\Gamma_M \cong \Gamma_S$ and M is isoclinic to S. Then $M \cong S$.*

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Characterization of Finite Groups by Non-Solvable Graphs and Solvabilizers

B. Akbari

Abstract

The non-solvable graph of a finite group G, denoted by $\mathcal{S}_G$, is a simple graph whose vertices are the elements of G and there is an edge between two elements $x, y \in G$ if and only if $\langle x, y\rangle$ is not solvable. If R is the solvable radical of G, the isolated vertices in $\mathcal{S}_G$ are exactly the elements of R. Thus, in the case when G is a non-solvable group, it is wise to consider the induced subgraph over $G \setminus R$ which is denoted by $\widehat{\mathcal{S}_G}$. Let G be a finite group and $x \in G$. The solvabilizer of x with respect to G, denoted by $Sol_G(x)$, is the set $\{y \in G \mid \langle x, y\rangle \text{ is solvable}\}$. In this paper, we are going to study some properties of $\widehat{\mathcal{S}_G}$ and the structure of $Sol_G(x)$ for every $x \in G$, more precisely.

Keywords and phrases: non-solvable graph, sovabilizer, finite group.

2010 *Mathematics subject classification:* Primary: 22D15, 43A10; Secondary: 43A20, 46H25.

1. Introduction

All groups appearing here are finite. One of the most important method and interesting subject is to study finite groups by quantitative properties associated with them. There are a lot of ways to relate a quantitative property to a finite group. One of them is to consider some properties of the graphs associated with it. Let G be a group. The *non-solvable graph* $\mathcal{S}_G$ is a simple graph that constructs as follows. The vertex set is G and two distinct elements x and y are adjacent if and only if the subgroup $\langle x, y\rangle$ is not solvable.

In fact, Thompson's Theorem asserts that a group G is solvable if and only if $\langle x, y\rangle$ is solvable for every $x, y \in G$. Hence, $\mathcal{S}_G$ is an empty graph if and only if G is solvable. Therefore, we only study $\mathcal{S}_G$ if G is not solvable.

For two non-empty subsets A, B of G, we call $Sol_A(B)$ the *solvabilizer* of B with respect to A

which is the subset

$$\{a \in A \mid \langle a, b\rangle \text{ is solvable } \forall b \in B\}.$$

Note that $Sol_A(B)$ is not necessarily a subgroup of G. We put $Sol_A(x) := Sol_A(\{x\})$ and $Sol(G) := Sol_G(G)$. Let R be the solvable radical of G. In [2], it is obtained that $Sol(G) = R$. It is also clear that $Sol_A(x) = Sol_A(\langle x\rangle)$. We focus our attention on $Sol_G(x)$.
It is shown in [3] that $Sol_G(x)$ is the union of all solvable subgroups of G containing x. It is also proved that $Sol_G(x)$ is a disjoint union of some cosets of $\langle x\rangle$.
According to above, for every $x \in G$ we have

$$\deg(x) = |G| - |Sol_G(x)|,$$

where $\deg(x)$ is the degree of vertex x in $\mathcal{S}_G$.
It is obvious that every two elements of $Sol(G)$ are not adjacent in $\mathcal{S}_G$. On the other hand, as mentioned before, $Sol(G) = R$ where R is the solvable radical of G, which means that if x is an element of G such that for every $y \in G$, $\langle x, y\rangle$ is solvable, then $x \in R$. Therefore, for all $x \in G \setminus R$, there exists an element $y \in G \setminus R$ such that $\langle x, y\rangle$ is not solvable. So we can conclude that the elements of R are exactly the isolated vertices in $\mathcal{S}_G$. Hence, if G is a non-solvable group, then it is logical to consider the induced graph of $\mathcal{S}_G$ with respect to $G \setminus Sol(G)$ which is denoted by $\widehat{\mathcal{S}_G}$. It is seen that the degree of vertex $x \in G \setminus Sol(G)$ in $\mathcal{S}_G$ is equal to its degree in $\widehat{\mathcal{S}_G}$.
The non-solvable graph of a group can be generalized in the following way (see [1]).
Let G be a finite group. The non-nilpotent graph of G, which is denoted by $\mathcal{N}_G$, is a simple graph whose vertices are the elements of G and two vertices x, y are adjacent by an edge if and only if $\langle x, y\rangle$ is not nilpotent. The induced subgraph of $\mathcal{N}_G$ on $G \setminus nil(G)$, where $nil(G) = \{x \in G \mid \langle x, y\rangle \text{ is nilpotent for all } y \in G\}$, was introduced as $\widehat{\mathcal{N}_G}$. This graph was completely examined in [1]. It is clear that $\mathcal{S}_G$ (resp. $\widehat{\mathcal{S}_G}$) is a subgraph of $\mathcal{N}_G$ (resp. $\widehat{\mathcal{N}_G}$).
We are going to focus on non-solvable graph. In fact, we are interested in finding the structure of a group through some properties of its non-solvable graph. Many properties of this graph was studied in [3].
First, we state some results obtained in [3] which will help us for further investigations. We begin with a Theorem taken from [2].

Theorem 1.1. *Let G be a non-solvable group and $x, y \in G$ such that $x, y \notin Sol(G)$. Then there exists $z \in G$ such that $\langle x, z\rangle$ and $\langle y, z\rangle$ are not solvable.*

It follows from Theorem 1.1 that the non-solvable graph $\widehat{\mathcal{S}_G}$ is connected and its diameter is at most 2. After that, the following Lemma was proved in [3].

Lemma 1.2. ([3]) *Let G be a non-solvable group. Then $diam(\widehat{\mathcal{S}_G}) = 2$.*

Lemma 1.3. ([3]) *Let G be a group. Suppose that $N \lhd G$ such that $N \subseteq Sol(G)$ and $x, g \in G$. Then the following statements hold:*

(1) $Sol_{G/N}(xN) = Sol_G(x)/N$;

(2) $Sol_G(gxg^{-1}) = gSol_G(x)g^{-1}$;

(3) *If $A, B \subseteq G$ are two subsets such that $A \subseteq B$ and $x \in A$ is an element, then $Sol_A(x) \subseteq Sol_B(x)$.*

Lemma 1.4. ([3]) *Let G be a group and $x \in G$. Then we have:*

(1) *$|Sol_G(x)|$ is divisible by $|Sol(G)|$;*

(2) *$|Sol_G(x)|$ is divisible by $o(x)$ and $|C_G(x)|$.*

Lemma 1.5. ([3]) *Let G be a non-solvable group and $x \in G \setminus Sol(G)$. Moreover, assume that $n = |G| - |Sol(G)|$. Then the following hold:*

(1) $2o(x) \leqslant deg(x)$;

(2) $5 < deg(x) < n - 1$;

(3) *$deg(x)$ is not a prime.*

As mentioned before, for an element $x \in G$, $Sol_G(x)$ need not be a subgroup of G in general. If G is a group in which $Sol_G(x) \leqslant G$ for all $x \in G$, then G is called an S-group. The structure of an S-group is studied in [3]. In fact, the following Lemma is proved.

Lemma 1.6. *Let G be a group. Then G is solvable if and only if G is an S-group.*

2. Main Results

In this section, we first examine some graphic properties of $\widehat{\mathcal{S}_G}$ and then we study the structure of finite groups through their non-solvable graphs.

Lemma 2.1. *Let G be a non-solvable group. Then $\widehat{\mathcal{S}_G}$ is not a tree.*

Before stating the following Lemma, it is good to mention that a set of vertices of a graph is independent if the vertices are pairwise nonadjacent. The *independence number* $\alpha(\Gamma)$ of a graph Γ is the cardinality of a largest independent set of Γ. Here, we give a lower bound for the independence number of $\widehat{\mathcal{S}_G}$ which is denoted by $\alpha(\widehat{\mathcal{S}_G})$. For the sake of simplicity of the notation we put $\alpha(G) = \alpha(\widehat{\mathcal{S}_G})$.

Lemma 2.2. *Let G be a non-solvable group. Then $\alpha(G) \geqslant max\{o(x) | x \in G\}$.*

Lemma 2.3. *Let G be a non-solvable group. Let H and N be two subgroups of G such that $N \lhd G$ and $N \subseteq Sol(G)$. Then the following statements hold:*

(1) *If x and y are joined in $\widehat{\mathcal{S}_H}$ for every $x, y \in H$, then x and y are joined in $\widehat{\mathcal{S}_G}$. In other words, $\widehat{\mathcal{S}_H}$ is a subgraph of $\widehat{\mathcal{S}_G}$.*

(2) *For two elements $x, y \notin Sol(G)$, xN and yN are adjacent in $\widehat{\mathcal{S}_{G/N}}$ if and only if x and y are adjacent in $\widehat{\mathcal{S}_G}$.*

Lemma 2.4. *Let G be a non-solvable group. Let H be a proper subgroup of G and $N \lhd G$ such that $N \subseteq Sol(G)$. Then the following statements hold:*

(1) *$\widehat{\mathcal{S}_H}$ is not isomorphic to $\widehat{\mathcal{S}_G}$.*

(2) *$\widehat{\mathcal{S}_{G/N}}$ is not isomorphic to $\widehat{\mathcal{S}_G}$.*

Here, we examine the structure of the solvabilizer of an element $x \in G$ with respect to G.

Theorem 2.5. *Let G be a non-solvable group and x be an element of G. Then $N_G(\langle x \rangle) \subseteq Sol_G(x)$. In particular, if $x, y \in G \setminus Sol(G)$ are two elements such that $y \in N_G(\langle x \rangle)$, then y is not adjacent to x in $\widehat{\mathcal{S}_G}$.*

A local subgroup of a group G is a subgroup K of G if there is a nontrivial solvable subgroup H of G such that $K = N_G(H)$.
When we are considering the solvibilizers of the elements belonging to the solvable subgroups of a finite group, we can generalize Theorem 2.5 to the following Lemma.

Lemma 2.6. *Let G be a group and $K = N_G(H)$ be a local subgroup of group G for a solvable subgroup H of G. Then for every $x \in H$, $K \subseteq Sol_G(x)$.*

Lemma 2.7. *Let G be a group. Then the local subgroup $K = N_G(H)$ of G is solvable if and only if $K \subseteq Sol_G(x)$ for every $x \in K$.*

In the following Theorem, we show that for all $x \in G$ where G is a non-solvable group, $deg(x) \neq n - 2$. Thus, we can conclude form Lemma 1.5 (2) that $deg(x) \leqslant n - 3$ for every $x \in G$.

Theorem 2.8. *Let G be a non-solvable group and $n = |G| - |Sol(G)|$. Then there is no element $x \in G \setminus Sol(G)$ such that $deg(x) = n - 2$.*

As mentioned before, for an element $x \in G$, $Sol_G(x)$ need not be a subgroup of G in general. In the following Lemma, we examine the structure of G in a case when $Sol_G(x)$ is a subgroup.

Lemma 2.9. *Let G be a non-solvable group and x be an element of G with $|Sol_G(x)| = p$ where p is a prime. Then G is a simple group.*

As a straightforward consequence of Lemma 2.9, we can show that a finite group G with a certain order and the following property is completely determined: For some prime p dividing $|G|$, G has an element x such that $|Sol_G(x)| = p$. So we can state the following Corollary.

Corollary 2.10. *Let G be a non-solvable group such that for some prime p dividing G, there exists an element $x \in G$ with $|Sol_G(x)| = p$ and*

$$G \notin \{O_{2n+1}(q), S_{2n}(q) : n \geqslant 3 \text{ and } q \text{ is odd }\} \cup \{A_3(2) \cong \mathbb{A}_8, A_2(4)\}.$$

If H is a finite group with $|H| = |G|$ and there exists a prime r dividing $|H|$ such that H has an element y where $|Sol_H(y)| = r$, then $H \cong G$.

For a finite group G, we define $\mathrm{Ord}(Sol_G) = \{|Sol_G(x)| \mid x \in G\}$. Now, it can be asked the following question.
Problem Let G and H be two finite groups. If $\mathrm{Ord}(Sol_G)$ coincides with $\mathrm{Ord}(Sol_H)$, then is G isomorphic to H?

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A generalisation of Feit-Seitz's theorem

Seyed Hassan Alavi

Abstract

In 1988, Feit and Seitz [3] studied composition factors of rational groups and determined all possible rational finite simple groups. They prove that such a simple group is isomorphic to $\mathrm{PSp}_6(2)$ or $\mathrm{P}\Omega_8^+(2)$. The main aim of this talk is to generalise this result to semi-rational groups.

Keywords and phrases: Rational groups, semi-rational groups, finite simple groups.

2010 *Mathematics subject classification:* Primary: 20D05; Secondary: 20E45.

1. Introduction

Let G be a finite group. An element $x \in G$ is said to be *semi-rational* in G if all generators of $\langle x \rangle$ lie in the union of two conjugacy classes of G, namely, the conjugacy class containing x, or the conjugacy class x^m of G, for some positive integer m. If each element of G is semi-rational in G, then G is called a *semi-rational* group. An special case of semi-rational groups are rational groups (or $\mathbb{Q}$-groups) in which the generators of $\langle x \rangle$, for all $x \in G$, are conjugate in G.

David Chillag and Silvio Dolfi [2] introduced the notion of semi-rational group. They studied solvable semi-rational groups and proved that if G is a semi-rational finite solvable group, then the set $\pi(G)$ of primes dividing the order of G is contained in $\{2, 3, 5, 7, 13, 17\}$. Chilag and Dolfi generalised Gow's result [4] for rational solvable groups which states that the only possible primes dividing the order of a rational solvable group are 2, 3, 5 or 7. In [1], semi-rational Frobenius groups have been studied and possible structure of complement and kernel of such groups have been determined. In particular, in the case where G is a semi-rational solvable Frobenius group, it is shown that $|\pi(G)| \leqslant 5$. This gives an answer to Problem 2 in [2] for

Table 1. Semirational finite simple groups.

Type	G
Cyclic	C_n where $n = 2, 3$
Lie	$\mathrm{L}_2(q)$, for $q = 7, 11$, $\mathrm{L}_3(4)$
	$\mathrm{U}_3(3)$, $\mathrm{U}_3(5)$, $\mathrm{U}_4(3)$, $\mathrm{U}_5(2)$, $\mathrm{U}_6(2)$
	$\mathrm{PSp}_4(3)$, $\mathrm{PSp}_6(2)$, $\mathrm{PSp}_6(3)$, $\mathrm{PSp}_8(2)$
	$\mathrm{P\Omega}_7(3)$, $\mathrm{P\Omega}_8^+(2)$, $\mathrm{P\Omega}_8^+(3)$
	${}^2\mathrm{E}_6(2)$, $\mathrm{F}_4(2)$, $\mathrm{G}_2(3)$, $\mathrm{G}_2(4)$
Sporadic	M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, HS, McL, He, Suz, Co_1, Co_2, Co_3,
	Fi_{22}, Fi_{23}, Fi_{24}', HN, Th, $\mathbb{B}$, $\mathbb{M}$.

semi-rational Frobenius groups.

2. Main Results

The main aim of this talk deals with studying semi-rational non-solvable groups. In particular, we are interested in finite simple groups. In 1988, Feit and Seitz [3] studied composition factors of rational groups and determined all possible rational simple groups, namely, $\mathrm{PSp}_6(2)$ or $\mathrm{P\Omega}_8^+(2)$. Here we generalise their result to semi-rational groups and prove

Theorem 2.1. *Let G be a finite simple group. Then G is a semi-rational group if and only if G is isomorphic to an alternating group* A_n *or one of the groups listed in* Table 1.

Note in passing that if G is a semi-rational non-solvable Frobenius group, then $|\pi(G)| \leqslant 4$, see [1]. As an immediate consequence of Theorem 2.1, we can answer Problem 2 in [2] for semi-rational finite simple groups:

Corollary 2.2. *Let G be a semi-rational finite simple group. If G is not an alternating group, then* $|\pi(G)| \leqslant 15$. *In particular, if G is a finite simple group of Lie type, then* $|\pi(G)| \leqslant 8$.

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Which group theoretic properties of a permutation group are inherited by its closures?

M. Arezoomand* and A. Abdollahi

Abstract

Given a permutation group G on a finite set Ω, for $k \geq 2$, the k-closure $G^{(k)}$ of G is the largest subgroup of $\mathrm{Sym}(\Omega)$ with the same orbits as G on the set Ω^k of k-tuples from Ω. Then $G \leq \ldots \leq G^{(k)} \leq G^{(k-1)} \leq \ldots \leq G^{(2)}$. In this paper, we review some basic properties of k-closures of permutation groups and report some group theoretic properties of a permutation group which are inherited by its closures. We prove that

(1) G is abelian of exponent e if and only if $G^{(2)}$ is abelian of exponent e.

(2) G is a p-group, p a prime, if and only if $G^{(2)}$ is a p-group.

(3) G is nilpotent if and only if $G^{(2)}$ is nilpotent.

Keywords and phrases: Permutation group, k-closure, nilpotent group.

2010 *Mathematics subject classification:* Primary: 20B05, 20D15; Secondary: 20F18.

1. Introduction

Given a permutation group G on a finite set Ω, for $k \geq 2$, the k-closure $G^{(k)}$ of G is the largest subgroup of $\mathrm{Sym}(\Omega)$ with the same orbits as G on the set Ω^k of k-tuples from Ω. The notion of 2-closure as a tool in the study of permutation groups was introduced by I. Schur [7]. The study of 2-closures of permutation groups has been initiated by Wielandt [7] in 1969, to present a unified treatment of finite and infinite permutation groups, based on invariant relations and invariant functions. For further studies and applications see [1–6]. In this paper, we review some basic properties of k-closures of permutation groups and report some group theoretic properties of a permutation group which are inherited by its closures. We prove that a permutation group

* speaker

G on a finite set Ω is nilpotent (abelian of exponent e, p-group) if and only if its 2-closure of Ω is.

2. Main Results

Recall that the k-closure of a permutation group G on a finite set Ω, $G^{(k)}$, is the largest subgroup of $\mathrm{Sym}(\Omega)$ with the same orbits as G on the set Ω^k of k-tuples from Ω. Let us start with an easy and important theorem:

Theorem 2.1. *[7, Theorem 5.6] Let $G \leq \mathrm{Sym}(\Omega)$. Then $s \in G^{(k)}$ if and only if for all $\alpha_1, \ldots, \alpha_k \in \Omega$ there exists $g \in G$ such that $\alpha_i^s = \alpha_i^g$ for all $i = 1, \ldots, k$.*

To prove our main results, we need the following definition and next lemmas:

Definition 2.2. *A permutation group G on Ω is called k-closed if $G = G^{(k)}$.*

Lemma 2.3. *[1, Lemma 2.2] Let G be a permutation group on Ω and H be a permutation group on Γ. If G and H are permutation isomorphic then $G^{(2)}$ and $H^{(2)}$ are permutation isomorphic.*

Lemma 2.4. *Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive p-group. Then $G^{(2)}$ is a p-group.*

In the following theorem, we prove that the 2-closure of any finite p-group is a finite p-group.

Theorem 2.5. *Let $G \leq \mathrm{Sym}(\Omega)$ and p be a prime. Then G is a p-group if and only if $G^{(2)}$ is a p-group.*

Lemma 2.6. *[1, Lemma 4.5] Let $G = H \times K \leq \mathrm{Sym}(\Omega)$ be transitive and $\Omega = \alpha^G$. If $(|H|, |K|) = 1$, then the action of G on Ω is equivalent to the action of G on $\Omega_1 \times \Omega_2$, where $\Omega_1 = \alpha^H$, $\Omega_2 = \alpha^K$ and G acts on $\Omega_1 \times \Omega_2$ by the rule $(\alpha^h, \alpha^k)^g = (\alpha^{hh_1}, \alpha^{kk_1})$, where $g = h_1 k_1$.*

Using above lemma, we prove the main result of the paper.

Theorem 2.7. *Let G be a finite permutation group on a set Ω. Then G is a nilpotent group if and only if $G^{(2)}$ is a nilpotent group.*

Note that the above theorem is not true for solvable groups. So one of the main questions in the study of 2-closures of permutation groups is the following open problem:

Question. For which solvable groups their 2-closure are solvable?

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On M-autocenter and M-autocommutator of group G

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Abstract

Let G be a group and $\mathrm{Aut}(G)$ be the full automorphisms group of G. The absolute center $L(G)$ of a group G is the subgroup of all elements fixed by every automorphism of G. We know that if $G/L(G)$ is finite then so is $\mathrm{Aut}(G)$. In this paper we introduce a new concept of certain subgroup of G, which is in a way a generalized version of autocenter and prove some results in this regard.

Keywords and phrases: Automorphisms group, autocenter, autocommutator.

2010 *Mathematics subject classification:* Primary: 20E45, 20B30; Secondary: 05A05, 04A16.

1. Introduction

Throughout this paper the following notation is used. For a group G, by G' and $Z(G)$, we denote the commutator subgroup and the center of G, respectively.
For any group H and abelian group K, $\mathrm{Hom}(H, K)$ denotes the group of all homomorphisms from H to K, where $(f.g)(x) = f(x)g(x)$ for all $f, g \in \mathrm{Hom}(H, K)$ and $x \in G$.
Let G be a group and $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$ denote the group of all automorphisms and inner automorphisms of G, respectively. An automorphism α of G is called central if $x^{-1}\alpha(x) \in Z(G)$ for all $x \in G$. The set of all central automorphisms of G, denoted by $\mathrm{Aut}_c(G)$, is a normal subgroup of $\mathrm{Aut}(G)$. Notice that the elements of $\mathrm{Aut}_c(G)$ act trivially on G'.
Let M and N be two normal subgroups of G. By $\mathrm{Aut}^M(G)$, we mean the subgroup of $\mathrm{Aut}(G)$ consisting of all the automorphisms which centralize G/M and by $\mathrm{Aut}_N(G)$, we mean the subgroup of $\mathrm{Aut}(G)$ consisting of all the automorphisms which centralize N. We denote $\mathrm{Aut}^M(G) \cap \mathrm{Aut}_N(G)$ by $\mathrm{Aut}_N^M(G)$.

If $x, y \in G$, then x^y denotes the conjugate element $y^{-1}xy \in G$. The commutator of two elements $x, y \in G$ is defined by $[x, y] = x^{-1}x^y$ and more generally, the autocommutator of the element $g \in G$ and the automorphism $\alpha \in \mathrm{Aut}(G)$ is defined to be $[g, \alpha] = g^{-1}g^{\alpha} = g^{-1}\alpha(g)$.
Many mathematicians got interested to study the structure of certain subgroups of an automorphism group. For a given group G with some properties, it would be interesting to determine the relationship between finiteness certain subgroup of G and some subgroup of $\mathrm{Aut}(G)$.
A classical result of Schur (1904) states that for any group G, if $G/Z(G)$ is finite then the commutator subgroup G is also finite.
Consider the following definition of $Z(G)$ and G':

$$Z(G) = \{x \in G : [g, \alpha] = 1, \forall \alpha \in \mathrm{Inn}(G)\},$$

$$G' = \langle [g, \alpha] : g \in G, \alpha \in \mathrm{Inn}(G) \rangle.$$

In 1994, Hegarty defined the subgroups $L(G)$ and $K(G)$ of G as follows:

$$L(G) = \{x \in G : [g, \alpha] = 1, \forall \alpha \in \mathrm{Aut}(G)\},$$

$$K(G) = \langle [g, \alpha] : g \in G, \alpha \in \mathrm{Aut}(G) \rangle.$$

$L(G)$ and $K(G)$ are called autocenter and autocommutator of G, respectively. One notes that, $L(G)$ and $K(G)$ are characteristic subgroups of G. Hegarty showed that if $G/L(G)$ is finite, then $K(G)$ and the automorphism group $\mathrm{Aut}(G)$ are both finite. We recall that he used the notation G^* for $K(G)$.
In 1955, Haimo introduced the following subgroup of a given group G, which we denote it by $L_c(G)$,

$$L_c(G) = \{x \in G : [x, \alpha] = 1, \forall \alpha \in \mathrm{Aut}_c(G)\}.$$

In 2016 in [1], Davoudirad and coauthor proved that if $G/L_c(G)$ is finite, then so is $\mathrm{Aut}_c(G)$. Our main motivation behind this work is extend the concept of the autocenter and the autocommutator of G and we obtain some interesting results. Notice that as a consequence of our main results, we derived the main results of [1] and [4], in this regard.

2. Main Results

First, we make the following definitions.

Definition 2.1. *Let G be a group and M be a normal subgroup of G. We define*

$$L_M(G) = \{x \in G : [x, \alpha] = 1, \ \forall \alpha \in Aut^M(G)\},$$

$$K_M(G) = \langle [g, \alpha] : g \in G, \alpha \in Aut^M(G) \rangle,$$

where $L_M(G)$ and $K_M(G)$ are called M-autocenter and M-autocommutator of G, respectively.

It is easy to check that $L_M(G)$ and $K_M(G)$ are characteristic subgroups of G. Moreover $L_M(G)$ contains $L(G)$ and $K_M(G)$ is contained in $K(G)$.

In the following results, $L_M(G)$ is assumed to be central.

Definition 2.2. *Let G be a group and M is a normal subgroup of G. We define two subgroups of $Aut(G)$ and G as follow:*

$$C_M(G) = C_{Aut_M(G)}(Aut^M(G)) = \{\alpha \in Aut_M(G) : \alpha\beta = \beta\alpha \ \forall\beta \in Aut^M(G)\},$$

$$E_M(G) = [G, C_M(G)].$$

Clearly, by the definition of $L_M(G)$, $\mathrm{Aut}^M(G)$ acts trivially on the M-autocenter of G. The following lemma gives the important property of $E_M(G)$. Note $K_M(G)$ does not carry over such a property.

Proposition 2.3. *Let G be a group and M be a normal subgroup of G where $[E_M(G), M] = 1$. Then $Aut^M(G)$ acts trivially on the subgroup $E_M(G)$.*

It is fairly easy to deduced the following corollary from Proposition 2.3.

Corollary 2.4. *Let G be a group and M be a normal subgroup of G where $[E_M(G), M] = 1$. Then $E_M(G) \leq L_M(G)$.*

Also we have the following results about $\mathrm{Aut}^M(G)$.

Proposition 2.5. *Let G be a group such that M is contained in $L_M(G)$. Then $Aut^M(G) \simeq Hom(G/L_M(G), M)$*

Let $M^* = M \cap L_M(G)$, then we have

$$\mathrm{Aut}^{M^*}(G) = \{\alpha \in \mathrm{Aut}^M(G) : [x, \alpha] \in L_M(G) \ \forall x \in G\},$$

is a normal subgroup of $\mathrm{Aut}^M(G)$.

Proposition 2.6. *Let G be a group and M be a normal subgroup of G, if $G/L_M(G)$ is finite, then so is $Aut^M(G)/Aut^{M^*}(G)$.*

By using the above proposition, we conclude the following proposition.

Proposition 2.7. *Let G be a group and M be a normal subgroup of G where the factor group $G/L_M(G)$ is finite. Then the following assertions are equivalent:*

(i) $K_M(G)$ *is finite;*

(ii) $Aut^M(G)$ *is finite;*

(iii) $Aut^{M^*}(G)$ *is finite.*

We next pose the problem: "Let M be a normal subgroup of G. what conditions on G is sufficient to ensure that $\mathrm{Aut}^M(G)$ is finite?"

Theorem 2.8. *Let G be a group and M be a normal subgroup of G. If $G/L_M(G)$ is finite, then so are $K_M(G)$ and $Aut^M(G)$.*

We remark that the converse of Theorem 2.8 is not true. In [2], Fournelle construct an infinite group G for which $K(G)$ is finite, but $G/L(G)$ is infinite. So in this group we have $K_M(G)$ is also finite and $G/L_M(G)$ is infinite.
However, the converse of Theorem 2.8 remains true when $\mathrm{Aut}^M(G)$ is finite. In fact we have the following theorem.

Theorem 2.9. *Let G be a group and M be a normal subgroup of G. If $K_M(G)$ and $Aut^M(G)$ are both finite, then so is $G/L_M(G)$*

In particular, Theorem 2.8 and Theorem 2.9 are interest for $M = G$. In fact, as immediate consequences of these theorems, we obtain the main results of Hegarty in [4].

Corollary 2.10. *([4,* Theorem*]) If $G/L(G)$ is finite then so are $K(G)$ and $Aut(G)$.*

Corollary 2.11. *([4,* Corollary*]) If $K(G)$ and $Aut(G)$ are both finite, then so is $G/L(G)$.*

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On commuting automorphisms of certain groups

N. Azimi Shahrabi* and M. Akhavan-Malayeri

Abstract

Let G be a group. An automorphism α of a group G is called a commuting automorphism if each element g in G commutes with its image $\alpha(g)$ under α. Let M be a characteristic subgroup of G. If the set of all commuting automorphisms of qoutent group G/M is a group, then we give some sufficient conditions on G such that the set of all commuting automorphisms of group G is a subgroup of $Aut(G)$.

Keywords and phrases: commuting automorphism, p-group, finitely generated group.

2010 *Mathematics subject classification:* Primary: 20F28; Secondary 20E36, 20E28.

1. Introduction

Let G be a group. By $Aut(G)$ and $Z(G)$ we denote the group of all automorphisms and the center of G, respectively. An automorphism α of G is called a commuting automorphism if $g\alpha(g) = \alpha(g)g$ for all $g \in G$. The set of all commuting automorphisms of the group G is denoted by $A(G)$. These automorphisms were first studied for various classes of rings [1]. The following problem was proposed by I. N. Herstein to the American Mathematical Monthly: If G is a simple non-abelian group, then $A(G) = 1$ [4]. Giving answer to Hersteins problem, Laffey in 1998 [5], proved that $A(G) = 1$ provided G has no non-trivial abelian normal subgroups. Also, Pettet gave a more general statement proving that $A(G) = 1$ if $Z(G) = 1$ and the commutator subgroup of G is equal to G (see [5]). In 2002, Deaconescu, Silberberg and Walls proved a number of interesting properties of commuting automorphisms [2], and raised the following natural question about $A(G)$: Is it true the set $A(G)$ is always a subgroup of $Aut(G)$? They

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themselves answered the question in negative by constructing an extra special group of order 2^5.

Definition 1.1. *A group G is called $A(G)$-group if the set*

$$A(G) = \{\alpha \in Aut(G) : g\alpha(g) = \alpha(g)g \ \ for\ all\ g \in G\}$$

forms a subgroup of $Aut(G)$.

Vosooghpour and Akhavan-Malayeri [7] showed that for a given prime p, minimum order of a non-$A(G)$ p-group G is p^5. They proved that there exists a non-$A(G)$ p-group G of order p^n for all $n \geq 5$. They also showed that if G is a nilpotent group of maximal class, then G is an $A(G)$-group.

Fouladi and Orfi showed that, if G is a finite AC-group or a p-group of maximal class or a metacyclic p-group, then G is an $A(G)$-group [3].

In 2015 Rai proved that a finite p-group G of coclass 2, for an odd prime p, is an $A(G)$-group.

2. Main Results

Suppuse M is a characteristic subgroup of G, such that $\overline{G} = G/M$ is an $A(\overline{G})$-group, then G is not necessarily an $A(G)$-group. For example, let p be an odd prime number. There exists a non-$A(G)$ extra special p-groups such that $\overline{G} = G/M$ is an $A(\overline{G})$-group and $A(\overline{G}) = Aut(\overline{G})$ [7].

Let M be a characteristic subgroup of G. If the set of all commuting automorphisms of qoutent group G/M is a group, then we give some sufficient conditions on G such that the set of all commuting automorphisms of group G is a subgroup of $Aut(G)$.

Remark: It is well known that if G is finitely generated and $n \in \mathbb{N}$, then the number of subgroups H of G such that $|G : H| = n$ is finite. Let these be $H_1, H_2, ..., H_t$. It is clear that $\bigcap_{i=1}^{t} H_i$ is a characterstic subgroup of G. We denote it by $Ch_n(G)$.

The main results of this paper are as follows:

Theorem 2.1. *Let G be a finitely generated group and $H \leqslant G$. If*

i) $|G : H| = n < \infty$.

ii) $G' \cap H = 1$.

iii) $G/Ch_n(G) = \overline{G}$ is an $A(\overline{G})$-group.

Then G is an $A(G)$-group.

Theorem 2.2. *Let G be a central by finite group such that $Z(G)$ is torsion-free. If $\overline{G} = G/Z(G)$ is an $A(\overline{G})$-group, then G is an $A(G)$-group.*

Theorem 2.3. *Let G be a finite group such that it has a central p-sylow subgroup. If $\overline{G} = G/P$ is an $A(\overline{G})$-group, then G is an $A(G)$-group.*

We have the following direct consequence of the above theorem.

Corollary 2.4. *Let G be a finite nilpotent group such that it has an abelian p-sylow subgroup P. If $\overline{G} = G/P$ is an $A(\overline{G})$-group, then G is an $A(G)$-group.*

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Quasi-permutation representations of some finite p-groups

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Abstract

For a finite group G, we denote by $p(G)$ the minimal degree of faithful permutation representations of G, and denote by $q(G)$ and $c(G)$, the minimal degree of faithful representation of G by quasi-permutation matrices over the rational field $\mathbb{Q}$ and the complex field $\mathbb{C}$, respectively. In this paper, we calculate $p(G)$, $q(G)$ and $c(G)$ for the groups of order p^5, where p is an odd prime.

Keywords and phrases: Quasi-permutation representation, p-groups, irreducible character .

2010 *Mathematics subject classification:* Primary: 22D15, 43A10; Secondary: 43A20, 46H25.

1. Introduction

By a classical theorem of Cayley, each group can be represented as a group of permutations of some set. In this field of study, one often needs to take the permuted set as small as possible. Thus the *minimal faithful permutation degree* $p(G)$ of a finite group G is defined as the least positive integer n such that G is isomorphic to a subgroup of the symmetric group S_n (or to a group of permutation matrices). Various interesting results have been obtained about $p(G)$. For example, in [2], it has been shown that if $A = A_1 \times \cdots \times A_r$ is an abelian group with each A_i cyclic of prime power order a_i, then $p(G) = a_1 + \cdots + a_r$. More generally, if H and K are nontrivial nilpotent groups, then $p(H \times K) = p(H) + p(K)$, which emphasizes the importance of studying $p(G)$ for finite p-groups (see [7]).

In a parallal direction, one may define two other degrees corresponding to embeddings of

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a finite group G in special types of matrix groups. In this way, we obtain two other degrees $q(G)$ and $c(G)$ which can be completely determined from the character table of G and often give best possible lower bounds for $p(G)$. In order to deal with them, we introduce the notion of a quasi-permutation matrix. By a *quasi-permutation matrix* over a subfield F of complex field $\mathbb{C}$, we simply mean a square matrix over F with non-negative integral trace. For a finite group G, let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field $\mathbb{Q}$, and let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices. Notice that every permutation matrix is a quasi-permutation matrix. Evidently, we have $c(G) \leqslant q(G) \leqslant p(G)$.

The quantities $q(G)$ and $c(G)$ were introduced in [1] and have been studied in [2], [3], etc. For example, in [2] and [3], $c(G)$, $q(G)$ and $p(G)$ were calculated for abelian and metacyclic groups. In fact, using the above notation we have $c(A) = q(A) = a_1 + \cdots + a_r - n$ for an abelian group A, where n is the largest integer such that C_6^n is a direct summand of A. If G is a p-group, then we have $n = 0$ and $p(G) = c(A) = q(A) = a_1 + \cdots + a_r$. In [1], it has been shown that for a finite p-group G of class 2 with cyclic centre, $c(G) = q(G) = p(G) = |Z(G)||G : Z(G)|^{1/2}$. Moreover, in [4], it has been shown that if the Schur indices of all complex irreducible characters over $\mathbb{Q}$ of a finite p-group G are equal to 1 (especially this is true when p is an odd prime), then $c(G) = q(G) = p(G)$. In [5], $c(G)$, $q(G)$ and $p(G)$ were calculated for the groups of order p^4. In this paper, we calculate $c(G)$, $q(G)$ and $p(G)$ for the groups of order p^5, where p is an odd prime. Notice that the groups of order p^5 are determined by H. A. Bender in [6].

2. Main Results

Definition 2.1. *Let G be a finite group. Let χ be an irreducible complex character of G. Then define*

(1) $d(\chi) = |\Gamma(\chi)|\chi(1)$.

(2) $$m(\chi) = \begin{cases} 0 & \text{if } \chi = 1_G \\ \left|\min\{\sum_{\alpha\in\Gamma(\chi)} \chi^\alpha(g) : g \in G\}\right| & \textit{otherwise} \end{cases},$$

(3) $c(\chi) = \sum_{\alpha\in\Gamma(\chi)} \chi^\alpha + m(\chi)1_G$.

Lemma 2.2. *Let G be a p-group whose center $Z(G)$ is minimally generated by d elements. Let $c(G) = \xi(1)+m(\xi)$ and $\xi = \sum_{i\in I} \Psi_i$. Let Ψ_i's satisfy in the conditions of the algorithm $c(G)$. Then*

(1) $m(\xi) = \frac{1}{p-1} \sum_{i \in I} \Psi_i(1)$,

(2) $|I| = d$.

The same assertions is valid for $q(G)$.

Lemma 2.3. *Let G be a finite non-abelian p-group of order p^n with p odd and* $\mathrm{cd}(G) = \{1, p\}$, *where $n \geq 5$. Assume that G has center of order p^{n-2}. Then $|G'| = p$.*

Theorem 2.4. *Let G be a finite non-abelian p-group of order p^n with p odd and* $\mathrm{cd}(G) = \{1, p\}$, *where $n \geq 5$. Assume that G has cyclic center of order p^{n-2}. Then $c(G) = q(G) = p(G) = p^{n-1}$.*

Theorem 2.5. *Let G be a finite p-group of order p^n, where $n \geq 5$. Assume that G has a non-cyclic center of order p^{n-2} and $d(G) = 2$ with $G = \langle x, y \rangle$ and $\langle x \rangle \cap \langle y \rangle = 1$, where $o(x) = p^{n-2}$, $o(y) = p^{n-3}$. Then $|G'| = p$ and $c(G) = q(G) = p(G) = p^{n-2} + p^{n-3}$.*

Theorem 2.6. *Let G be a finite p-group of order p^5. Also let $Z(G) \cong C_{p^2} \times C_p$ and $d(G) = 3$. Then $c(G) = q(G) = p(G) = 2p^2$, $p^3 + p$ or $p^3 + p^2$.*

Theorem 2.7. *Let G be a finite p-group of order p^5. Suppose that $|Z(G)| = p^3$ and $d(G) = 4$. Then $c(G) = q(G) = p(G) = p^2 + 2p$, $2p^2 + p$, $p^3 + p$ or $2p^2$.*

Theorem 2.8. *Let G be a finite non-abelian p-group of order p^5. Let G has cyclic center of order p^2. Then $p(G) = q(G) = c(G) = p^3$ or p^4.*

Theorem 2.9. *Let G be a finite p-group of order p^5. Also let $Z(G) \cong C_p \times C_p$. Then $c(G) = q(G) = p(G) = p^2 + p$, $2p^2$, $p^3 + p$ or $p^3 + p^2$.*

Theorem 2.10. *Let G be a finite p-group of order p^5. Also let $Z(G) \cong C_p$. Then $c(G) = q(G) = p(G) = p^2$ and p^3.*

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Unitary groups and symmetric designs

Ashraf Daneshkhah

Abstract

The main part of this talk is devoted to studying almost simple groups with socle projective special unitary groups acting flag-transitively as automorphism groups of block designs.

Keywords and phrases: Automorphisms groups, symmetric designs, flag-transitive designs.

2010 *Mathematics subject classification:* Primary: 20B25; Secondary: 05B0, 05B25.

1. Introduction

A t-design $\mathcal{D}$ with parameter (v, k, λ) or a t-(v, k, λ) design is a point-line rank 2 geometry whose point set $\mathcal{P}$ is of size v and lines are k-subsets of $\mathcal{P}$ known as *blocks* such that each t-subset of $\mathcal{P}$ is contained in exactly in λ blocks. A design is called *symmetric* if the number of points and blocks are equal, in other words, points and blocks play the same role. *Biplanes* and *triplanes* are symmetric 2-designs with $\lambda = 2$ and $\lambda = 3$, respectively. It is known that for a t-(v, k, λ) design and any positive integer s such that $1 < s \leqslant t$, there also exists a 2-(v, k, λ') for some λ'. Thus one may focus on studying 2-designs. A group of *automorphisms* of a design consists of permutations of points mapping blocks to blocks and preserving the incidence relation. An automorphism group G of $\mathcal{D}$ is called *flag-transitive* if it is transitive on the set of flags of $\mathcal{D}$. If G is primitive on the point set $\mathcal{P}$, then G is said to be *point-primitive*. In this talk, we give a survey on recent studies on flag-transitive automorphisms groups of symmetric designs.

2. Main results

In addition to the fundamental study of flag-transitive regular linear spaces [4], there have been numerous contributions to the study of flag-transitive and point-primitive designs with small λ, in particular, biplanes and triplanes. In 1961, D. G. Higman and J. E. McLaughlin [7] in their considerably influential paper proved that flag-transitivity implies point-primitivity in linear space. Then Kantor [8] classified flag-transitive symmetric $(v, k, 1)$ designs (projective planes) of order n. In 1990 a deep result, namely the classification flag-transitive finite linear spaces relying on the Classification of Finite Simple Groups (CFSG) was announced in [4]. Although, flag-transitive biplanes are not point-primitive, Regueiro [9] proved that an flag-transitive and point-primitive automorphism group of biplanes is of almost simple or affine type, and so using CFSG, she determined all flag-transitive and point-primitive biplanes. In conclusion, she gave a classification of flag-transitive biplanes except for the 1-dimensional affine case. Thereafter, Zhou and Dong studied and classified triplanes with flag-transitive and point-primitive automorphism groups except for the 1-dimensional affine case.

Recently, Tian and Zhou proved that an automorphism group acting flag-transitively and point-primitively on symmetric 2-designs with $\lambda \leqslant 100$ must be of almost simple or affine type [10] and based on their computational evidence they conjectured that the same assertion holds for any flag-transitive and point-primitive automorphism group of a symmetric 2-design. It is worth noting that this conjecture is already proved for 2-designs when $(r, \lambda) = 1$ [6, 13]. Therefore, it is interesting to study such designs whose socle is of almost simple type or affine type with large λ. In this direction, it is recently shown in [1] that there are only five possible symmetric (v, k, λ) designs admitting a flag-transitive and point-primitive automorphism group G satisfying $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$ where $X = PSL_2(q)$, see also [12]. This study for $X := PSL_3(q)$ gives rise to one nontrivial design (up to isomorphism) which is a Desarguesian projective plane $PG_2(q)$ and $PSL_3(q) \leqslant G$ see [2].

Theorem 2.1. *(Alavi-Bayat-Daneshkhah) Suppose that $\mathcal{D}$ is a symmetric (v, k, λ) design with $\lambda \geqslant 1$ admitting a flag-transitive and point-primitive automorphism group G of almost simple type with socle X. Then*

(a) *if $X = PSL_2(q)$, then $\mathcal{D}$ is of parameter $(7, 3, 1)$, $(7, 4, 2)$, $(11, 6, 3)$, $(15, 8, 4)$ or $(11, 5, 2)$ respectively for $q = 7$, 7, 11, 11 or 9;*

(b) *if $X = PSL_3(q)$, then $\mathcal{D}$ is a Desarguesian projective plane $PG_2(q)$ and $PSL_3(q) \leqslant G$;*

However, $X = PSU_3(q)$ gives rise to no non-trivial flag-transitive symmetric designs for $q \geqslant 4$, see [5].

Theorem 2.2. *(Daneshkhah-Zang Zarin) Suppose that $\mathcal{D}$ is a symmetric (v, k, λ) design with $\lambda \geqslant 1$ admitting a flag-transitive and point-primitive automorphism group G of almost simple type with socle $X = PSU_3(q)$. Then $\mathcal{D}$ is a Menon design of parameter $(36, 15, 6)$ and $G = PSU_3(3)$ or $PSU_3(3) : 2$.*

In the case where X is a sporadic simple group, there exist four possible parameters, see [11]. The same problem has been investigated for X being an exceptional finite simple group of Lie type, see [3].

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A numerical invariant for finite groups

H. R. Dorbidi

Abstract
For a finite group G define $m(G) = \frac{|G'|+|Z(G)|}{|G|}$. We prove some results about the range of this function. Also we determine the structure of $G/Z(G)$ for for some special values of $m(G)$.

Keywords and phrases: Commutator subgroup, Center of a group .
2010 *Mathematics subject classification:* Primary: 20D99.

1. Introduction

Let G be a finite group. Two important subgroups of G are $Z(G)$ the center of G and G' the commutator subgroup. The structure of G influenced by the structure of this subgroup. For example G' determines the solvability of G and $G/Z(G)$ determines nilpotency of G. In this paper we define a numerical invariant and study some properties of it. For a finite group G define $m(G) = \frac{|G'|+|Z(G)|}{|G|} = \frac{1}{[G:Z(G)]} + \frac{1}{[G:G']}$. It is clear that $m(G) \leq 2$ and $m(G) = 2$ if and only if G is the trivial group. Let $R = \{m(G) : G \text{ is a finite group}\}$ be the range of the function m. We prove some results about the set R. Also we determine the structure of $G/Z(G)$ for for some special values of $m(G)$.

Throughout this paper G is a finite group. The group G is called a perfect group if $G = G'$. The dihedral group of order $2n$ is denoted by D_n.

2. Main Results

Definition 2.1. *Let p be a prime number. Define a function $\psi(p^a) = (p^a - 1)\cdots(p - 1)$ and extend it multiplicatively to $\mathbb{N}$.*

Definition 2.2. *A natural number n is called a nilpotent(abelian,cyclic) number if every group of order n is a nilpotent(abelian,cyclic) group.*

Theorem 2.3. *[1]*

1. *n is a nilpotent number if and only if $(\psi(n), n) = 1$.*
2. *n is an abelian number if and only if $(\psi(n), n) = 1$ and n is a cube-free number.*
3. *n is a cyclic number if and only if $(\psi(n), n) = 1$ and n is a square-free number(Note that for a square free number $\phi(n) = \psi(n)$).*

Let G be a finite group and $[G : Z(G)] = |G/Z(G)| = s, [G : G'] = |G/G'| = t$. Then $m(G) = \frac{1}{s} + \frac{1}{t}$.

Remark 2.4. 1. *If G is a non-abelian group then s is not a cyclic number i.e $(s, \phi(s)) > 1$.*
2. *If s is an abelian number then $G' \subseteq Z(G)$ and $s|t$.*

Theorem 2.5. *Let $q = \frac{a}{b}$ be a positive rational number. The equation $\frac{1}{m} + \frac{1}{n} = \frac{a}{b}$ has a solution if and only if b^2 has a divisor d such that $a|b + d, a|\frac{b(b+d)}{d}$.*

Proof. Assume $\frac{1}{m} + \frac{1}{n} = \frac{a}{b}$. Then $amn = bm + bn$. Hence $(am - b)(an - b) = a^2mn - abm - abn + b^2 = b^2$. If $d = am - b$ then $d|b^2$ and $a|b + d, a|\frac{b(b+d)}{d}$. Conversely, if $b + d = am$ and $an = \frac{b(b+d)}{d}$ then $\frac{1}{m} + \frac{1}{n} = \frac{a}{b}$. □

Example 2.6. *The equation $\frac{1}{m} + \frac{1}{n} = \frac{3}{7}$ has no solution. So $m(G) \neq \frac{3}{7}$ for any group G i.e $\frac{3}{7} \notin R$.*

Example 2.7. 1. *If G is an abelian group of order n then $m(G) = 1 + \frac{1}{n}$.*
2. *If G is a simple group of order n then $m(G) = 1 + \frac{1}{n}$.*
3. *$m(S_n) = \frac{1}{2} + \frac{1}{n!}$ for $n \geq 3$.*
4. *If n is an odd number then $m(D_n) = \frac{n+1}{2n}$.*
5. *If n is an even number then $m(D_n) = \frac{n+4}{4n}$.*

Theorem 2.8. *The group G is an abelian group or a perfect group if and only if $m(G) > 1$. In particular $m(G) = 1 + \frac{1}{n}$ for some natural number n.*

Proof. Let $m(G) > 1$. If $s, t > 1$ then $\frac{1}{s} + \frac{1}{t} \leq 1$. So $s = 1$ or $t = 1$. Hence G is an abelian group or G is a perfect group. □

Theorem 2.9. *Assume $m(G) \leq 1$.*

1. *If $d = |G'Z(G)/Z(G)| = |(G/Z(G))'|$ then $d|s|dt$.*
2. *If $s = 4$ then $G' \subseteq Z(G)$. In particular $4|t$ and $m(G) \leq \frac{1}{2}$.*
3. *If $s \geq 6$ then $m(G) \leq \frac{2}{3}$.*

4. *If* $\frac{1}{2} < m(G)$ *then* $t = 2$.
5. $m(G) \notin (\frac{2}{3}, 1]$.
6. $m(G) \notin (\frac{5}{8}, \frac{2}{3})$.

PROOF. Since $m(G) \leq 1$, so $s, t > 1$.

1. It is clear that $\frac{s}{d}[G/Z(G) : G'Z(G)/Z(G)] = [G : G'Z(G)]|[G : G'] = t$. So $d|s|dt$.
2. Since $s = 4$ is an abelian number, So $G' \subseteq Z(G)$ and $4|t$ which implies that $m(G) \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
3. If $s \geq 6$ then $m(G) \leq \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$.
4. According to part (2), $s \geq 6$. So $t = 2$.
5. This is clear by parts (1), (2).
6. By part (4), $t = 2$. Hence $\frac{1}{s} < \frac{1}{6}$ which implies $s \geq 8$. So $m(G) \leq \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$.

□

Theorem 2.10. *$m(G) = \frac{1}{2}$ if and only if $G/Z(G) \cong \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ and $Z(G) = G'$. Moreover G is a $2-$group.*

PROOF. Since $m(G) = \frac{1}{s} + \frac{1}{t} = \frac{1}{2}$, so $s, t > 2$. If $s \geq 6$ then $t \leq 3$. So $t = 3$ and $s = 6$. Hence $G/Z(G) \cong S_3$. Thus $|G'Z(G)/Z(G)| = 3$. This implies that $2 = [G : G'Z(G)]|[G : G'] = 3$ which is a contradiction. So $s = 4$ which implies $t = 4$ and $Z(G) = G'$. Hence $G/Z(G) \cong \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$. □

Theorem 2.11. *$m(G) = \frac{2}{3}$ if and only if $G/Z(G) \cong S_3$ and $Z(G) \subseteq G'$. Moreover G is metabelian.*

PROOF. According to theorem 2.9, $s \geq 6$. So $t = 2$. Hence $s = 6$. So $G/Z(G) \cong S_3$. Thus $|G'Z(G)/Z(G)| = 3$. This implies that $2 = [G : G'Z(G)] = [G : G']$. Hence $G' = G'Z(G)$. So $Z(G) \subseteq G'$. Since $|G'/Z(G)| = 3$, so G' is abelian. The converse is clear. □

Theorem 2.12. *$m(G) = \frac{3}{5}$ if and only if $G/Z(G) \cong D_5$ and $Z(G) \subseteq G'$. Moreover G is metabelian.*

PROOF. According to theorem 2.9, $s \geq 6$. So $t = 2$. Hence $s = 10$. So $G/Z(G) \cong D_5$. Thus $|G'Z(G)/Z(G)| = 5$. This implies that $2 = [G : G'Z(G)] = [G : G']$. Hence $G' = G'Z(G)$. So $Z(G) \subseteq G'$. Since $|G'/Z(G)| = 5$ so G' is abelian. The converse is clear. □

Lemma 2.13. *Let A ba finite abelian group. Then $m(G \times A) = \frac{1}{s} + \frac{1}{t|A|}$.*

Assume $\frac{1}{m} + \frac{1}{n} \in R$. Is it true that $\frac{1}{m} + \frac{1}{nk} \in R$.

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The precise center of pre-crossed modules over a fixed base group

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Abstract

In this paper, the concept of the precise center of a group is generalized to the category of pre-crossed modules over a fixed base group. We also determine some relations between the precise center of pre-crossed modules and the non-abelian exterior square, the second homology of pre-crossed modules.

Keywords and phrases: pre-crossed module; precise center; non-abelian exterior square; Homology.

2010 *Mathematics subject classification:* Primary: 22D15, 43A10; Secondary: 43A20, 46H25.

1. Introduction

Let G, M be arbitrary groups. A *pre-crossed G-module* (M, ∂) is a group homomorphism $\partial : M \to G$, together with an action of G on M which is denoted by ${}^{g}m$ for any $g \in G$ and $m \in M$ satisfying $\partial({}^{g}m) = g\partial(m)g^{-1}$, for all $g \in G$, $m \in M$. (M, ∂) is called a crossed G-module if in addition it satisfies the Peiffer identity ${}^{\partial(m)}m' = mm'm^{-1}$ for every $m, m' \in M$. A subgroup N of M is called a *pre-crossed G-submodule* of M if N is stable under the action of G. A *morphism* $f : (M_1, \partial_1) \to (M_2, \partial_2)$ of pre-crossed G-modules is a commutative diagram

$$\begin{tikzcd} M_1 \arrow[rr, "f"] \arrow[dr, "\partial_1"'] & & M_2 \arrow[dl, "\partial_2"] \\ & G & \end{tikzcd}$$

such that f is an G-equivariant group homomorphism, i.e. $f({}^{g}m) = {}^{g}f(m)$ for all $m \in M_1$ and $g \in G$. In this paper, we denote by $\mathcal{PCM}(G)$ the category of pre-crossed G-modules, and by $\mathcal{CM}(G)$ the Birkhoff subcategory of crossed G-modules.

Let (M, ∂) be a pre-crossed G-module. The *Peiffer commutator* of two elements $m, m' \in M$ is defined by

$$\langle m, m' \rangle = mm'm^{-1} \, ({}^{\partial m}m')^{-1}.$$

The *Peiffer subgroup* of two pre-crossed submodules N_1, N_2 is the subgroup $\langle N_1, N_2 \rangle$ generated by all the Peiffer elements $\langle n_1, n_2 \rangle$ and $\langle n_2, n_1 \rangle$ with $n_1 \in N_1$ and $n_2 \in N_2$. For basic properties of these subgroups see [3].

In [3], Conduché and Ellis defined the second homology group $H_2(M)_G$ of a pre-crossed G-module (M, ∂) by means of a Hopf type formula as follows. Take a free presentation $1 \to R \to F \to M \to 1$ in the category $\mathcal{PCM}(G)$ of the pre-crossed G-modules and define

$$H_2(M)_G = \frac{R \cap \langle F, F \rangle}{\langle R, F \rangle},$$

Note that this homology coincides with the Baer invariants in the category $\mathcal{PCM}(G)$ relative to variety $\mathcal{CM}(G)$ of crossed modules.

Let (M, ∂) be a pre-crossed L-module, the *non-abelian exterior square* $M \wedge M$ is the group generated by the elements $m \wedge m'$ with $m, m' \in M$ subject to the following relations

$$m \wedge m'm'' = (m \wedge m')(m \wedge m'')(\langle m, m'' \rangle^{-1} \wedge^{\partial m} m'), \tag{1}$$

$$mm' \wedge m'' = (m \wedge m'm''m'-1)({}^{\partial m}m' \wedge {}^{\partial m}m''), \tag{2}$$

$$\langle m, m' \rangle \wedge \langle n, n' \rangle = (m \wedge m')(n \wedge n')(m \wedge m')^{-1}(n \wedge n')^{-1}, \tag{3}$$

$$(\langle m, m' \rangle \wedge m'')(m'' \wedge \langle m, m' \rangle) = (m \wedge m')({}^{\partial m''}m \wedge {}^{\partial m''}m')^{-1}, \tag{4}$$

$$k \wedge k = 1, \tag{5}$$

for all $m, m', m'', n, n' \in M$ and $k \in \ker \partial$. Note that G acts on $M \wedge M$ by ${}^{g}(m \wedge m') = {}^{g}m \wedge {}^{g}m'$. Also, there exists an equivariant homomorphism $\delta_M : M \wedge M \to M$ defined by $\delta_M(m \wedge m') = \langle m, m' \rangle$. We remark that by choosing $G = 1$, $\partial = 0$, the group $M \wedge M$ is the usual non-abelian exterior product of groups defined by R. Brown and J.-L. Loday [2].

2. Main Results

Definition 2.1. *The crossed-center of a pre-crossed G-module (M, ∂) is the submodule*

$$Z(M, \partial) = \{m \in \ker \partial \;:\; \langle m, M \rangle = \langle M, m \rangle = 1\}.$$

An epimorphism $(N, \delta) \overset{\pi}{\twoheadrightarrow} (M, \partial)$ of pre-crossed G-modules is called a crossed-central extension of (M, ∂) if $\langle \ker \pi, M \rangle = 1$.

Definition 2.2. *We define the precise center* $Z^*(M, \partial)$ *of a pre-crossed G-module* (M, ∂) *to be the intersection of submodules* $\pi(Z(N, \delta))$ *where* $\pi : (N, \delta) \twoheadrightarrow (M, \partial)$ *is a crossed-central extension of* (M, ∂).

Theorem 2.3. *Let* (M, ∂) *be a pre-crossed G-module and* $1 \to R \to F \xrightarrow{\pi} M \to 1$ *a free presentation of* (M, ∂) *in* $\mathcal{PCM}(G)$ *then*

$$Z^*(M, \partial) = \bar{\pi}(Z(\frac{F}{\langle F, R\rangle})),$$

where $\bar{\pi} : \frac{F}{\langle F,R\rangle} \to M$ *is the homomorphism induced by* π.

Using the concept of non-abelian exterior square, we can obtain a new description of the precise center of pre-crossed modules.

Theorem 2.4. *Let* (M, ∂) *be a pre-crossed G-module such that admits a free presentation* $1 \to R \to F \to M \to 1$ *in* $\mathcal{PCM}(G)$ *satisfying the homomorphism* $\delta_F : F \wedge F \to F$ *is injective, then*

$$Z^*(M, \partial) = \{m \in M \mid m \wedge n = 1, \text{ for all } n \in M\}.$$

Proposition 2.5. *Let* $(K, 0) \overset{i}{\rightarrowtail} (N, \delta) \overset{\pi}{\twoheadrightarrow} (M, \partial)$ *be a crossed-central extension of* (M, ∂). *Then there exists the following exact sequence of groups*

$$H_2(N)_G \to H_2(M)_G \to K \to \frac{N}{\langle N, N\rangle} \to \frac{M}{\langle M, M\rangle} \to 1.$$

In [3], it was shown that the natural map $H_2(N)_G \to H_2(M)_G$ can be surjective under some conditions. In the next theorem, we give a criterion for the injectivity of this map.

Theorem 2.6. *Let* $(K, 0) \overset{i}{\rightarrowtail} (N, \delta) \overset{\pi}{\twoheadrightarrow} (M, \partial)$ *be a crossed-central extension of* (M, ∂). *Then the following conditions are equivalent*

i) $i(K) \subseteq Z^*(N, \delta)$,

ii) The natural map $H_2(N)_G \to H_2(M)_G$ *is injective.*

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Divisible modulo its torsion group field

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Abstract

For some absolutely algebraic field F_0 of characteristic $p > 0$ and κ an infinite cardinal, it is shown that there exists a field F such that $F^* \cong F_0^* \oplus (\oplus_\kappa \mathbb{Q})$.

Keywords and phrases: Multiplicative group, Field, Divisible .

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1. Introduction

In (cf. [3, p.299]) L. Fuchs asks which abelian groups can be the multiplicative groups of fields? R. M. Dicker in ([2]) gives an answer to this question in terms of the existence of a certain function on the group with zero adjoined. This question is largely unsolved, though quite a few partial results have been obtained. An abelian group G (written additively) is divisible if for every $g \in G$ and every positive integer n, there exists $h \in G$ with $nh = g$. An abelian group G is divisible modulo its torsion group if $G/T(G)$ is divisible, where $T(G)$ is the group of torsion elements of G. The famous example of a divisible abelian group is the additive group $\mathbb{Q}$ of rational numbers, a torsion free divisible abelian group. Another example is the direct limit of the cyclic groups $\mathbb{Z}/\langle p^n \rangle (p$ a prime). This group is the so-called quasi-cyclic group of type p^∞ÂĄ, denoted $C(p^\infty)$, a torsion divisible abelian group. The structure of divisible abelian groups is well-understood as the following theorem from [5] shows:

* speaker

Theorem A. Let G be an abelian group. Then G is divisible if and only if G is a direct sum of copies of $\mathbb{Q}$ and $C(p^\infty)$ for various primes p.

Given a field F, denote by F^* the multiplicative group of F. For any prime p, let $\mathbb{F}_p$ be its prime subfield. An absolutely algebraic field, denoted by aaf, is an algebraic extension of $\mathbb{F}_p$. One may easily check that for any aaf F we have $F = \bigcup_{n\in S} \mathbb{F}_{p^n}$, where S is a nonempty subset of the positive integers such that for any $n, m \in S$ we have $\mathbb{F}_{p^{lcm(n,m)}} \subseteq F$. Also, if $n \in S$ and $x|n$, then $\mathbb{F}_{p^x} \subseteq F$. These conditions are necessary and sufficient conditions for when F is an absolutely algebraic field (aaf). It is also clear that any aaf is perfect.
Here we investigate the question of when a multiplicative group of a field is divisible modulo its torsion group. In this direction we have the following results from [1].

Theorem B. Let G be an abelian group with finite, nonzero torsion free rank. Then G is not isomorphic to the multiplicative group of any field.

Theorem C. Let G be a torsion-free divisible group of infinite rank and let p be an arbitrary prime integer. Then there is a field F of characteristic p such that F* is isomorphic to $\mathbb{F}_p^* \oplus G$.

For some absolutely algebraic field F_0 of characteristic $p > 0$ and κ an infinite cardinal, it is shown that there exists a field F such that $F^* \cong F_0^* \oplus (\oplus_\kappa \mathbb{Q})$.

2. Main Results

Before starting our main result, we the following famous theorem.

Abel's Theorem. (cf. [4, p.297]) Let K be a field, $n > 2$ an integer, and $a \in K$ with $a \neq 0$. Assume that for all prime numbers p such that $p|n$, we have $a \notin K^p$, and if $4|n$ then $a \notin -4K^4$. Then $X^n - a$ is irreducible in K[X].

Now, we are ready to prove our main result.

Theorem 2.1. *Let F be an absolutely algebraic field of characteristic $p > 0$ containing a primitive $q-$root of unity ω_q for some prime $q \neq p$. Then, for any natural number i, $\mathbb{F}_{p^{q^i}} \subseteq F$ if and only if the polynomial $x^{q^i} - \omega_q$ has a root in F(or splits over F) for any natural number i.*

Proof. First, let $\mathbb{F}_{p^{q^i}} \subseteq F$ for any natural number i. So, there is no field extension of degree q over F. Now, assume on the contrary that there exists a natural number n such that $x^{q^{n-1}} - \omega_q$

has a root in F but $x^{q^n} - \omega_q$ has no root in F. Since every finite subgroup of the multiplicative group of a field is cyclic, we conclude that the maximal q-group in F^* is a finite cyclic group with q^n elements. Take ω_{q^n}, a generator of this group, which is a primitive q^n-root of unity in F. This implies that there is no root in F for the polynomial $x^q - \omega_{q^n}$. But, by Abel's Theorem, F has an extension of degree q, which is a contradiction.

Conversely, assume that $x^{q^i} - \omega_q$ has a root in F for any natural number i. Let n be the least natural number such that $\omega_q \in \mathbb{F}_{p^n}$. By Fermat' Theorem, $q \nmid n$. Consider the maximal q-subgroup of $\mathbb{F}^*_{p^n}$ which is cyclic and take ω_{q^m}, a generator of this group, for some natural number m. Now, if $q \neq 2$, by Abel's Theorem, $x^{q^i} - \omega_{q^n}$ is irreducible over $\mathbb{F}_{p^n}$ for any natural number i. Let $a_i \in F$ be a root of this polynomial in some extension. Then, $[\mathbb{F}_{p^n}(a_i) : \mathbb{F}_{p^n}] = q^i$ and $\mathbb{F}_{p^n}(a_i) = \mathbb{F}_{p^{nq^i}}$ for any natural number i. Thus, $\mathbb{F}_{p^{q^i}} \subseteq \mathbb{F}_{p^n}(a_i) \subseteq F$, as desired. For the case $q = 2$ and $\sqrt{-1} \in \mathbb{F}_p$, it is easily checked that $a \notin -4\mathbb{F}_p^4$ and hence Abel's Theorem may be applied to obtain the result. Finally, if $q = 2$ and $\sqrt{-1} \notin \mathbb{F}_p$, then $\mathbb{F}_p(\sqrt{-1}) = \mathbb{F}_{p^2} \subseteq F$. The maximal 2-subgroup of $\mathbb{F}^*_{p^2}$ is then cyclic. Now, use the same argument as above to end the proof.

□

3. Examples

In this section we present some examples that may clarify the results mentioned earlier in the paper.

Example 3.1. *Let $f = \bigcup_{n\in S} \mathbb{F}_{2^n}$, where $S = \{n \mid (n, 2) = (n, 3) = 1\}$ and set $F = f \propto \mathbb{F}_{2^2}$ (composition of f and $\mathbb{F}_{2^2}$). We claim that F^* is a direct product of a divisible and a nontrivial bounded group. If $q \neq 3$ is a prime with $\omega_q \in F$, where ω_q is the primitive q-root of unity, then ω_q is in the divisible part of F^*. On the other hand, the bounded part of F^* is the maximal 3-group in F^* as claimed.*
Let $F_{n,m} = F \propto \mathbb{F}_{2^{2^n}} \propto \mathbb{F}_{2^{3^m}}$, where n, m are nonnegative integers. $[F_{n,m} : F] = 2^n 3^m$. By the same argument as used above, for any nonnegative integers n and m, $F^*_{n,m}$ is direct product of a divisible and a nontrivial bounded group. This means that any finite extension of F has this property. When K is a filed such that $T(K^*)$ is a direct product of a divisible and a nontrivial bounded group $K^* \cong T(K^*) \oplus (\oplus_\kappa \mathbb{Q})$, where κ is either zero or infinite cardinal. If $T(K^*) = F^*$, then K^* is a direct product of a divisible and a nontrivial bounded group. Also, for any finite extension L of K, L^* is a direct product of a divisible and a bounded group.

Example 3.2. *Set $f = \bigcup_{n\in S} \mathbb{F}_{p^n}$ with the property that for any prime q with $\mathbb{F}_{p^q} \subseteq f$ there exists* *some natural number n_q such that $\mathbb{F}_{p^{q^{n_q}}} \nsubseteq f$. Then the divisible part of f^* is a trivial group. Furthermore, any finite extension of f has our desired property. We note that these fields are qGf. Now, assume that κ is an arbitrary infinite cardinal and f is an absolutely algebraic field such that f is a qGf. Then, we may find a field F with F is qGf and $F^* \cong f^* \oplus (\oplus_\kappa \mathbb{Q})$.*

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Frattini Subgroup Of $GL_n(D)$ over real closed fields

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Abstract

Given a positive integer n, let $GL_n(D)$ be the general skew linear group over D with center the real closed field F. Then we have $Frat(GL_n(D)) \cong F^*$ for $n > 1$ and $F^* \cap Frat(GL_n(D)) \cong F^*$ for $n = 1$.

Keywords and phrases: Frattini subgroup, Division ring .

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1. Introduction

Let D be a division ring with center F. Given a positive integer n, denote by $A := M_n(D)$ the full $n \times n$ matrix ring over D, by $A^* := GL_n(D)$ the general skew linear group over D, and by $A' := SL_n(D)$ the derived group of A^*. When the dimension $[A : F]$ of A/F is finite, it is known (cf. [6, p. 44]) that the group $G(A) := A^*/F^*A'$ is torsion of a bounded exponent dividing the index of A. The existence of a maximal subgroup in D^* in the case $n = 1$ is an open question. We recall that the Frattini subgroup of a group G is defined as the intersection of all maximal subgroups of G. We denote the Frattini subgroup of group G by $\Phi(G)$. If we have no maximal subgroups in G, we set $\Phi(G) = G$. For finite groups G the Frattini subgroup, being a nilpotent subgroup, plays a remarkable role in dealing with the structure of G. As for infinite groups, in this direction we should mention a result of Wehrfritz which asserts that the Frattini subgroup of a finitely generated linear group is nilpotent. Here we investigate the Frattini subgroup of the general skew linear group $A^* = GL_n(D)$ for various division rings D. If $n > 1$, Lemma 1 of [3]

* speaker

shows that $\Phi(A^*)$ is central. But it is not known if the same assertion is valid for the case $n = 1$. In [3], it is shown that if A/F is finite dimensional, then $\Phi(F^*)Z(A') \subseteq G \subseteq (\cap_p F^{*p})Z(A')$, where the intersection is taken over all p such that $(p, [A : F]) = 1$, when $G = F^* \cap \Phi(A^*)$. In addition, if $G(A) = 1$, then $G = \Phi(F^*)Z(A')$.

Let D be a division ring with center F and G be a subgroup of $GL_n(D)$. We denote by $F[G]$ the F-linear hull of G, i.e., the F-algebra generated by elements of G over F. For any group G we denote its center by $Z(G)$. $[G : H]$ denotes the index of H in G, and $\langle H, K\rangle$ the group generated by H and K, where K is a subgroup of G. For a group G, we define $G^p := \{g^p \mid g \in G\}$. We shall identify the center FI of $M_n(D)$ with F. For a given ring R, the group of units of R is denoted by $R^* = U(R)$. We denote the derived group of a group G by G'. If A is an F-central simple algebra, we sometimes write $A \in Br(F)$, where $Br(F)$ denotes the Brauer group of F. $RD_{A/F}(A^*)$ is the group of reduced norm of A^* over F, $SK_1(A) = \{a \in A^* | RD_{A/F}(a) = 1\}/A'$, and $G(A) := A^*/F^*A'$. In this article we prove that for a given positive integer n, let $GL_n(D)$ be the general skew linear group over the radically real closed field F. Then we have $Frat(GL_n(D)) \cong F^*$ for $n > 1$ and $F^* \cap Frat(GL_n(D)) \cong F^*$ for $n = 1$.

2. Main Results

Before stating our next results, we recall the following theorems from [3].

Theorem A. Let A be a finite dimensional F-central simple algebra. Then, $\Phi(F^*)Z(A') \subseteq F^* \cap \Phi(A^*) \subseteq (\cap_p F^{*p})Z(A')$, where $(p, [A : F]) = 1$. In addition, if $G(A) = 1$, then $F^* \cap \Phi(A^*) = \Phi(F^*)Z(A')$.

Theorem B. Let D be a division ring with center F, and $n > 1$ be an integer. Then we have:
(i) $\Phi(GL_n(D)) \subset Z(GL_n(D))$.
(ii) $\Phi(GL_n(D)) = Z(GL_n(D))$ provided that F^* does not contain any maximal subgroup.

Theorem C. Let G be an abelian group. Then, we have $\Phi(G) = \cap_p G^p$, where p ranges over the set of all prime numbers.

Recall from the theory of ordered fields that a field F is said to be formally real if F admits an ordering, if and only if -1 is not a sum of squares in F. F is said to be real closed, if F is formally real and no proper algebraic extension of F is formally real. For example, the real number field $\mathbb{R}$ is real closed. If F is real closed, then $\sqrt{-1} \notin F$ and $F(\sqrt{-1})$ is algebraically closed. Conversely, every algebraically closed field E of characteristic zero contains a real closed subfield F such that $E = F(\sqrt{-1})$ (cf. [5]). According to Theorem 4.1.25 of [4], if F is a real closed field, then $F^* \cong \mathbb{Z}_2 \oplus (\oplus_{|F|}\mathbb{Q})$. F is said to be radically real closed, if F is indivisible and $F(\sqrt{-1})$ is divisible. As shown in [7], for such an F we have $\mathrm{Br}(F) = \mathbb{Z}_2$ and F^* is isomorphic to the unit group of a real closed field. It is clear that any real closed field is radically real closed.

We are now in a position to prove the main theorem of this article.

Theorem 2.1. *Given a positive integer n, let $GL_n(D)$ be the general skew linear group over the radically real closed field F. Then we have $\Phi(GL_n(D)) \cong F^*$ for $n > 1$ and $F^* \cap \Phi(GL_n(D)) \cong F^*$ for $n = 1$.*

PROOF. As shown in [7], for such an F we have $\mathrm{Br}(F) = \mathbb{Z}_2$ and F^* is isomorphic to the unit group of a real closed field. It is clear that any real closed field is radically real closed. On the other hand the only quaternion division algebra over a radically real closed field F is the ordinary quaternion division algebra. Which means if D is a division algebra over F then $D \cong (\frac{-1,-1}{F})$. Then D is the four dimensional algebra over F with basis $1, i, j, k$ and multiplication defined by $i^2 = j^2 = -1, k = ij = -ji$. F is not algebraically closed but the field extension $C = F(\sqrt{-1})$ is algebraically closed. Every quaternion element a can be expressed in the form $a = (a_1 + ia_2) + j(a_3 + ia_4)$, where each a_i is in F.

Set $A = M_n(D)$. First, we show that $Z(A') = \langle -1 \rangle$. Now, let $n = [A : F]$ and assume that $c \in Z(A') = A' \cap F^*$. Thus $Nrd_{A/F}(A^*)(c) = 1$. It is known (cf. [6, p. 44]) that the group $G(A) := A^*/F^*A'$ is torsion of a bounded exponent dividing the index of A. Therefore c is a torsion element. Thus $Z(A') \subseteq \langle -1 \rangle$. On the other hand $-1 = iji^{-1}j^{-1}$. Hence, $Z(A') = \langle -1 \rangle$, as we claimed.

Now, by Theorem A, we know that $\Phi(F^*)Z(A') \subseteq F^* \cap \Phi(A^*) \subseteq (\cap_p F^{*p})Z(A')$, where $(p, [A : F]) = 1$. But, for any radically real closed field F, we have $F^{*p} = F^*$ for any odd prime p. We have $F^* \cong \mathbb{Z}_2 \oplus (\oplus_{|F|}\mathbb{Q})$. Using Theorem C and Theorem B, we conclude that $\Phi(F^*) = F^{*p}$ for any odd prime p. Since $Z(A') = \langle -1 \rangle$, we conclude that $\Phi(GL_n(D)) \cong F^*$ for $n > 1$ and $F^* \cap \Phi(GL_n(D)) \cong F^*$ for $n = 1$.

□

Notice that when $n = 1$. By Example of [1], we know that $\mathbb{H}^* := GL_1(\mathbb{H})$ has a maximal $\mathbb{H}^*$, i.e., L is the intersection of all maximal subgroups of $\mathbb{H}^*$. It is clear that L is a normal subgroup of $\mathbb{H}^*$. By Theorem C, we know that $\mathbb{H}' \subset L$, i.e., $\mathbb{H}' \subset M$ for each maximal subgroup of $\mathbb{H}^*$. But this contradicts Example 1 of [1]. Therefore, we must have $L \subset \mathbb{R}^*$, where $\mathbb{R}$ is the field of real numbers. On the other hand, since $\mathbb{H}' \not\subset M$ for each maximal subgroup M of $\mathbb{H}^*$, by Proposition 1 of [2], we conclude that $\mathbb{R}^* \subset M$ for each maximal subgroup of $\mathbb{H}^*$. Thus, $Frat(\mathbb{H})^* = \mathbb{R}^*$.

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Finite groups with two composite character degrees

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Abstract

Let G be a finite group and let cd(G) be the set of all irreducible character degrees of G. We consider finite groups G with the property that cd(G) has at most two composite members. We derive a bound 6 for the size of character degree sets of such groups. There are examples in both solvable and nonsolvable groups where this bound is met. In the case of nonsolvable groups, we are able to determine the structure of such groups with $|\mathrm{cd(G)}| \leqslant 6$.

Keywords and phrases: finite group; irreducible character; composite degree .

2010 *Mathematics subject classification:* Primary:20C15 ; Secondary: 20D05 .

1. Introduction

Given a finite group G, let Irr(G) be the set of all complex irreducible characters of G and let cd(G) denote the set $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. The character degree set cd(G) is a key tool in investigations about G, which has been studied extensively by many people. These studies generally focus on answering the following two main questions: Which sets of positive integers can occur as cd(G) for some group G, and what can be said about the structure of those groups that have a set X as their character degree set?

To aid in the studying of these questions, we consider the groups which have a small number of composite (irreducible) character degrees. This property for character degrees of finite groups has been studied in [3–5]. In fact, Isaacs and Passman in [3, 4] characterized finite groups G with $\mathrm{cd}(G) \setminus \{1\}$ consisting of primes, and Y. J. Liu and Y. Liu [5] prove that if G is a solvable group with exactly one composite degree in cd(G), then either $|\mathrm{cd}(G)| \leqslant 4$ or $\mathrm{cd}(G) = \{1, p, q, f, qf\}$

* speaker

for some primes p, q, f, while if G is nonsolvable, then $G \cong A_5 \times A$, where A is an abelian group. In this paper, we consider finite groups which have two composite character degrees.

2. Main Results

Theorem 2.1. *Let G be a solvable group with at most two distinct composite character degrees. Then either $|\mathrm{cd}(G)| \leqslant 5$ or $\mathrm{cd}(G) = \{1, p, q, r, qr, a\}$, where p, q, r are distinct primes and a is a composite integer divisible by at least one of the primes p, q, r.*

Given a prime p, there always exists a finite group P with $\mathrm{cd}(P) = \{1, p\}$. Perhaps the easiest example is an extraspecial group of order p^3. See also [2, Theorem 12.11]. However, given primes q, r, there need not exist a finite group Q with $\mathrm{cd}(Q) = \{1, q, r\}$. In [4], Isaacs and Passman determined all primes q, r for which there exists a group Q with $\mathrm{cd}(Q) = \{1, q, r\}$. Examples of such groups Q are given in [4]. If we take $G = P \times Q$, then $\mathrm{cd}(G) = \{1, p, q, r, pq, pr\}$. Hence, we have examples of solvable groups with two distinct composite character degrees and $|\mathrm{cd}(G)| = 6$. There are solvable groups G with $|\mathrm{cd}(G)| = 5$ or 4 and $\mathrm{cd}(G)$ contains exactly two composite degrees. Examples 18.8(b) and (c) of [1] show that there are groups of order $48p^2$ (p a prime) such that $\mathrm{dl}(G) = 5$ and

$$\mathrm{cd}(G) = \{1, 2, 3, 4, 48\} \text{ or } \mathrm{cd}(G) = \{1, 2, 3, 4, 24\}.$$

Also, by Example 27.11(a) of [1], there exists a 2-transitive group B_1 of degree 3^4 and order $2^4 \cdot 3^4 \cdot 5$ such that $\mathrm{cd}(B_1) = \{1, 4, 5, 80\}$ and $\mathrm{dl}(B_1) = 4$.

We next obtain a classification of nonsolvable groups with at most two composite character degrees. It turns out that there are only a few types of sets with at most two composite members that can occur as $\mathrm{cd}(G)$ for some nonsolvable group G.

Theorem 2.2. *Let G be a nonsolvable group with at most two distinct composite character degrees. Then one of the following holds:*

(i) $G \cong \mathrm{PSL}_2(2^f) \times A$, *where $f \geqslant 2$, A is an abelian group and at least one of the numbers $2^f \pm 1$ is prime. In this case,* $\mathrm{cd}(G) = \{1, 2^f - 1, 2^f, 2^f + 1\}$.

(ii) $G \cong \mathrm{PSL}_2(\mathrm{p}) \times A$, *where $p > 5$ and $\frac{p+(-1)^{(p-1)/2}}{2}$ are primes, A is an abelian group. Here,* $\mathrm{cd}(G) = \{1, \frac{p+(-1)^{(p-1)/2}}{2}, p, p-1, p+1\}$.

(iii) $G/Z(G) \cong \mathrm{PGL}_2(\mathrm{p})$, $G' \cong \mathrm{PSL}_2(p)$ *or* $\mathrm{SL}_2(p)$, *where $p \geqslant 5$ is a prime and $C_G(G') = Z(G)$. In this case,* $\mathrm{cd}(G) = \{1, p-1, p, p+1\}$.

(iv) $G = G'Z(G)$ *and* $G' \cong \mathrm{SL}_2(5)$. *Here,* $\mathrm{cd}(G) = \{1, 2, 3, 4, 5, 6\}$.

(v) $G/L \cong \mathrm{PSL}_2(4)$ *and* $\mathrm{cd}(G) = \{1, 3, 4, 5, 15\}$, *where L is the solvable radical of G.*

We remark that the structure of the groups given in Theorem 2.2(v) are determined in [7, Theorem B]. As an example of such groups, we mention from [6], the semidirect product of $\mathrm{PSL}_2(4) \cong \mathrm{SL}_2(4)$ acting on its natural module.

Combining Theorems 2.1 and 2.2, we obtain the following result.

Corollary 2.3. *If G is any group with two composite character degrees, then* $|\mathrm{cd}(G)| \leqslant 6$.

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Some properties of 2-Baer Lie algebras

Maryam Ghezelsoflo* and Mohammad Amin Rostamyari

Abstract

A Lie algebra L is said to be 2-Baer if for every one dimensional subalgebra K of L, there exists an ideal I of L such that K is an ideal of I.

In this talk, we study 2-Baer Lie algebras and also three classes of Lie algebras with 2-subideal subalgebras and give some relations among them.

Keywords and phrases: 2-subideal subalgebras; 2-Baer Lie algebra; nilpotent Lie algebra .

2010 *Mathematics subject classification:* Primary: 17B45; 17B30, Secondary: 17B99.

1. Introduction

Let K be a subalgebra of a Lie algebra L. We call K is n-subideal of L and denoted by $K \lhd_n L$, if there exist distinct subalgebras $K_1, K_2, \cdots, K_n$ such that

$$K \lhd K_1 \lhd K_2 \lhd \cdots \lhd K_n = L,$$

for some $n \in \mathbb{N}$.

In the present talk, we introduce a new notion of n-Baer Lie algebras and it is shown that some of the known results of n-Baer groups can be proved in n-Baer Lie algebras. A group G is called *n-Baer group* if all of its cyclic subgroups are n-subnormal.

A natural set up for Lie algebras is as following.

Definition 1.1. *A Lie algebra L is called n-Baer Lie algebra if all of its one dimensional subalgebras are n-subideal.*

* speaker

Clearly in a nilpotent group of class n, all subgroups are n-subnormal.

Conversely, by the well-known result of Roseblade [5], a group with all subgroups n-subnormal is nilpotent of class bounded by a function of n.

The next lemma describes the closure properties of the class of n-Baer Lie algebras.

Lemma 1.2. *Every subalgebra K of an n-Baer Lie algebra L is also n-Baer.*

2. Main Results

In this section we prove some structural results for 1 and 2-Baer Lie algebras.

Definition 2.1. *A Lie algebra L is called* 1*-Baer or Dedekind Lie algebra if all of its one dimensional subalgebras are ideal in L.*

We remind that a group in which all of its subgroups are normal, called *Dedekind group*. Such finite groups classified by Dedekind in 1897 [3], and the infinite case by Baer in [1]. Dedekind groups are either abelian or the direct product of the Quaternion group of order 8 by a periodic abelian group with no elements of order 4.

One can easily see that every abelian Lie algebra is Dedekind.

Let L be a Lie algebra over a field F of characteristic 0 and let $D(L)$ be the derivation algebra of L, that is, the Lie algebra of all derivations of L. For any non-zero element x of L, the linear transformation by the rule $ad_x : y \mapsto [x, y]$ is a derivation of L which is called *adjoint representation* or *inner derivation* and denote the set of all inner derivations of L by $\mathrm{Inn}(L)$.

Now, using the above discussion we state an important property of the Dedekind Lie algebras.

Proposition 2.2. *Let L be Dedekind Lie algebra, then L is nilpotent of class at most* 2.

The following lemma is interesting and useful property for 2-Baer Lie algebras.

Lemma 2.3. *Let H be a one dimensional subalgebra of a Lie algebra L. Then L is* 2*-Baer Lie algebra if and only if $[L, h, h] \subseteq H$, for all non-zero elements h of H.*

Recall that a linear transformation of a Lie algebra L is nilpotent if its n-th power is zero for some natural number n, i.e. $(ad_x)^n = 0$. Clearly $ad_x(y) = [x, y]$ and inductively

$$(ad_x)^2 y = ad_x \circ ad_x(y) = ad_x([x, y]) = [x, [x, y]],$$

$$(ad_x)^3 y = ad_x \circ ad_x \circ ad_x(y) = [x, [x, [x, y]]], \cdots .$$

By the definition of nilpotent Lie algebra one observes that if L is nilpotent, then ad_x is also nilpotent, for all $x \in L$. The converse is also true by Engel's theorem, which says that a finite dimensional Lie algebra L is nilpotent if and only if ad_x is nilpotent, for all x in L.

Considering the above discussion and Lemma 2.3, we have the following result.

Theorem 2.4. *Every* 2*-Baer Lie algebra is* 3*-Engel.*

Corollary 2.5. *If L is finite dimensional* 2*-Baer Lie algebra, then L is nilpotent.*

We denote the class of all 2-Baer Lie algebras, the class of Lie algebras in which every abelian subalgebra is 2-subideal, and the class of all Lie algebras in which every subalgebra is 2-subideal by $\mathcal{L}_B$, $\mathcal{L}_A$ and $\mathcal{L}_S$, respectively. It is obvious that $\mathcal{L}_S \subseteq \mathcal{L}_A \subseteq \mathcal{L}_B$.

In the rest of this talk we show that, for 2-dimensional Lie algebras the properties $\mathcal{L}_B$, $\mathcal{L}_A$ and $\mathcal{L}_S$ are equivalent.

The next result and Lemma 2.3, play important role in proving our main theorem.

Proposition 2.6. *Let C be a class of Lie algebras, which is closed under forming subalgebras. If $K \in C$ is a subalgebra of Lie algebra L, then K is* 2*-subideal in L if and only if $[L, x, y] \subseteq \langle x, y\rangle$, for all $x, y \in K$.*

The following corollary is an immediate consequence of the above proposition.

Corollary 2.7. *Let L be a Lie algebra. Then*
(*i*) *$L \in \mathcal{L}_S$ if and only if $[l, x, y] \subseteq \langle x, y\rangle$, for all $x, y, l \in L$;*
(*ii*) *$L \in \mathcal{L}_A$ if and only if $[l, x, y] \subseteq \langle x, y\rangle$, for all $x, y, l \in L$, with $[x, y] = 0$.*

Recall that $\mathcal{L}_S \subseteq \mathcal{L}_A \subseteq \mathcal{L}_B$. The following main theorem, gives the exact relations between $\mathcal{L}_S$, $\mathcal{L}_A$ and $\mathcal{L}_B$.

Theorem 2.8. *Let L be a Lie algebra over a fixed field F of characteristic $\neq 2$. Then $L \in \mathcal{L}_B$ is equivalent to $L \in \mathcal{L}_A$.*

Here, our interest in a generalization of n-Baer Lie algebras, in particular 2-Baer Lie algebras, is motivated by D. Cappitt and L.C. Kappe's researches in this aria in group theory (see [2] and [4] for more details).

For any Lie algebra L, let $T_n(L) = \langle x \in L : \langle x\rangle \ntriangleleft_n L\rangle$. If all one dimensional subalgebras are n-subideal in L, i.e. L is an n-Baer Lie algebra, we define $T_n(L) = 1$.

Definition 2.9. *If L is a Lie algebra with $T_n(L) \neq L$, then L is called a generalized n-Baer Lie algebra, and if in addition $T_n(L)$ is non-trivial, then L is called a generalized T_n-Lie algebra.*

The class of generalized n-Baer Lie algebras and the class of generalized T_n-Lie algebras will be denoted by $\mathcal{GB}_n$ and $\mathcal{GT}_n$, respectively.

Here, we provide some structure results for generalized 2-Baer Lie algebras.

Lemma 2.10. *Let* $L \in \mathcal{GB}_2$.
(i) *If* $x \in L \setminus T_2(L)$ *and* K *is a subalgebra of* L *containing* x, *then* $K \in \mathcal{GB}_2$.
(ii) *If* I *is an ideal of* L *contained in* $T_2(L)$, *then* $L/I \in \mathcal{GB}_2$.

Using the above lemma we have the following

Theorem 2.11. *Let* $L \in \mathcal{GB}_2$ *and* $x \in L \setminus T_2(L)$. *Then* $[L, x]$ *is nilpotent of class at most* 2. *In particular,* x *is a left* 3-*Engel element.*

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On the structure of non-abelian tensor square of p-groups of order p^4

T. J. Ghorbanzadeh*, M. Parvizi and P. Niroomand

Abstract

This paper is based on the class of p-groups of order p^4, we obtain $\nabla(G)$, $J_2(G)$, the non-abelian exterior square, the non-abelian tensor square, the tensor center and the exterior center of such groups.

Keywords and phrases: The non-abelian groups of order p^4, The non-abelian tensor square, The non-abelian exterior square .

1. Introduction

The non-abelian tensor square $G \otimes G$ of a group G is the group generated by the symbols $g \otimes h$ with defining relations

$$g_1 g \otimes h = ({}^{g_1}g \otimes {}^{g_1}h)(g_1 \otimes h) \text{ and } g \otimes hh_1 = (g \otimes h)({}^{h}g \otimes {}^{h}h_1)$$

for all $g, g_1, h, h_1 \in G$, where G acts on itself via conjugation.
The group $G \wedge G = G \otimes G/\nabla(G)$ is called non-abelian exterior square of G, where $\nabla(G) = \langle g \otimes g | g \in G \rangle$.
The maps $\kappa : G \otimes G \to [G, G]$ and $\kappa' : G \wedge G \to G'$ are epimorphisms and the kernel of κ is denoted by $J_2(G)$.
Brown and Loday in [3] describe the role of $J_2(G)$ in algebraic topology, they showed the third homotopy group of suspension of an Eilenberg-MacLane space $k(G, 1)$ satisfied the condition $\Pi_3(SK(G, 1)) \cong J_2(G)$. Given an abelian group A, from [6], $\Gamma(A)$ is used to denote the abelian

* speaker

group with generators $\gamma(a)$, for $a \in A$,
by defining relations
(i). $\gamma(a^{-1}) = \gamma(a)$.
(ii). $\gamma(abc)\gamma(a)\gamma(b)\gamma(c) = \gamma(ab)\gamma(bc)\gamma(ca)$.
for all $a, b, c \in A$. From [3], we have

Theorem 1.1. *Let G and H be abelian groups. Then*
(i) $\Gamma(G \times H) = \Gamma(G) \times \Gamma(H) \times (G \otimes H)$,
(ii)

$$\Gamma(\mathbb{Z}_n) = \begin{cases} \mathbb{Z}_n & \textit{if } n \textit{ is odd}, \\ \mathbb{Z}_{2n} & \textit{if } n \textit{ is even}. \end{cases}$$

Recall from [4], the concept of tensor and exterior center respectively, $Z^{\otimes}(G) = \{g \in G | g \otimes g_1 = 1_{G\otimes G}, \text{ for all } g_1 \in G\}$ and $Z^{\wedge}(G) = \{g \in G | g \wedge g_1 = 1_{G\wedge G}, \text{ for all } g_1 \in G\}$. A group G is called capable if there exist a group H such that $G \cong H/Z(H)$. The epicenter of G which is denoted by $Z^*(G)$ is defined as follows

Definition 1.2. *Let $\psi : E \to G$ be an arbitrary surjective homomorphism with $\ker \psi \subseteq Z(G)$. Then the intersection of all subgroups of the form $\psi(Z(G))$ is denoted by $Z^*(G)$.*

$Z^*(G)$ has the property that G is capable if and only if $Z^*(G) = 1$. Beyl and Tappe proved $Z^*(G)$ is isomorphic to $Z^{\wedge}(G)$, though $Z^*(G)$ is defined in a different fashion. The next lemma shows the relation between $Z^*(G)$, $Z^{\otimes}(G)$ and $Z^{\wedge}(G)$, which plays important role in this paper

Lemma 1.3. *[1,* p. 208*] and [4,* Propositin 16*] Let G be any group. Then $Z^*(G) = Z^{\wedge}(G)$ and $Z^{\otimes}(G) \leq Z^{\wedge}(G)$.*

Here we give the presentation of all p-groups of order p^4 as follows.

Theorem 1.4. *[5,* Theorem1.11*] Let G be a group of order p^4, where p is a prime. Then G is isomorphic to exactly one of the following groups .*

$$G_1 \cong \mathbb{Z}_{p^4},\ \ G_2 \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p,\ \ G_3 \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2},\ \ G_4 \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p,$$
$$G_5 \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p,\ \ G_6 \cong \mathbb{Z}_p \times E_1,\ \ G_7 \cong \mathbb{Z}_p \times E_2,$$
$$G_8 \cong \langle x, y, z | x^p = y^p = z^{p^2} = 1, [x, z] = [y, z] = 1, [x, y] = z^p \rangle,$$
$$G_9 \cong \langle x, y | x^{p^3} = y^p = 1, x^y = x^{1+p^2} \rangle,$$
$$G_{10} \cong \langle x, y | x^{p^2} = y^p = 1, [x, y, x] = [x, y, y] = 1 \rangle,$$
$$G_{11} \cong \langle x, y | x^{p^2} = y^{p^2} = 1, [x, y, x] = [x, y, y] = 1, [x, y] = x^p \rangle,$$
$$G_{12} \cong \langle x, y | x^4 = y^4 = 1, [x, y, x] = [x, y, y] = 1, [x, y] = x^2 y^2 \rangle,$$
$$G_{13} \cong \langle x, y | x^2 = y^8 = (xy)^2 = 1 \rangle,$$
$$G_{14} \cong \langle x, y | x^4 = y^2 = (xy)^2 \rangle,$$
$$G_{15} \cong \langle x, y | x^2 = 1, xyx = y^3 \rangle,$$
$$G_{16} \cong \langle x, y | x^{p^2} = y^p = 1, [x, y, x] = 1, [x, y, y] = x^p, [x, y, y, y] = 1 \rangle,$$
$$G_{17} \cong \langle x, y | x^{p^2} = y^p = 1, [x, y, x] = 1, [x, y, y] = x^{np}, [x, y, y, y] = 1 \rangle,$$
$$G_{18} \cong \langle x, y | x^9 = 1, x^3 = y^3, [x, y, x] = 1, [x, y, y] = x^6, [x, y, y, y] = 1 \rangle,$$
$$G_{19} \cong \langle x, y | x^p = 1, y^p = 1, [x, y, x] = [x, y, y, x] = [x, y, y, y] = 1 \rangle,$$
$$G_{20} \cong \langle x, y | x^p = 1, y^p = [x, y, y], [x, y, x] = [x, y, y, x] = [x, y, y, y] = 1 \rangle.$$

2. Main Results

In this section, we intend to obtain the structure of $G \wedge G, Z^{\wedge}(G), G \otimes G, Z^{\otimes}(G), \nabla(G)$ and $\Pi_3(K(G, 1))$ of non-abelian groups of order p^4 by using their presentation. By the following theorem we can obtain the structure of tensor square of groups with respect to the direct product of two groups.

Theorem 2.1. *[3,* Proposition 11*] Let G and H be groups. Then* $(G \times H) \otimes (G \times H) = (G \otimes G) \times (G^{ab} \otimes H^{ab}) \times (H^{ab} \otimes G^{ab}) \times (H \otimes H)$.

The next theorem, give us the structure of non-abelian tensor square of a group G with G^{ab} finitely generated as follows.

Theorem 2.2. *[2,* Corollary 1.4*] Let G be a group such that G^{ab} is finitely generated. If G^{ab} has no element of order two or G' has a complement in G, then $G \otimes G = \Gamma(G^{ab}) \times G \wedge G$.*

The key tool to obtain the structure of $G \wedge G$ is the following theorem.

Theorem 2.3. *[4, Proposition 16 (iv)] Let G be a group and $N \trianglelefteq G$. Then $G/N \wedge G/N \cong G \wedge G$ if and only if $N \leq Z^{\wedge}(G)$.*

Theorem 2.4. *Let G be a group of order p^4, where p is an odd prime. Then*

$$G \wedge G \cong \begin{cases} \mathbb{Z}_p & \text{if } G \cong G_9. \\ \mathbb{Z}_p^{(3)} & \text{if } G \cong G_7, G_8, G_{10}, G_{16}, G_{17}, G_{20}. \\ \mathbb{Z}_{p^2} & \text{if } G \cong G_{11}. \\ \mathbb{Z}_p^{(5)} & \text{if } G \cong G_6. \\ \mathbb{Z}_p^{(4)} & \text{if } G \cong G_{19}. \\ \mathbb{Z}_3^{(3)} & \text{if } G \cong G_{18}. \end{cases}$$

The next result is the non-abelian tensor square of the non-abelian groups of order p^4 of nilpotency class 3.

Theorem 2.5. *Let G be a group of order p^4, where p is an odd prime. Then*

$$G \otimes G \cong \begin{cases} \mathbb{Z}_p^{(6)} & \text{if } G \cong G_{16}, G_{17} \text{ or } G_{20}. \\ \mathbb{Z}_3^{(6)} & \text{if } G \cong G_{18}. \\ \mathbb{Z}_p^{(7)} & \text{if } G \cong G_{19}. \end{cases}$$

Following theorem is an important tool for the next investigations.

Theorem 2.6. *[4, Proposition16 (v)] Let G be a group and $N \trianglelefteq G$. Then $G/N \otimes G/N \cong G \otimes G$ if and only if $N \leq Z^{\otimes}(G)$.*

By [5] , we have

Lemma 2.7. *Let G be a group of order p^4, where p is prime. Then*

$$Z^{\wedge}(G) \cong \begin{cases} 1 & \text{if } G \cong G_6, G_{11}, G_{12}, G_{13} \text{ or } G_{19}. \\ \mathbb{Z}_p & \text{if } G \cong G_7, G_8, G_{10}, G_{16}, G_{17} \text{ or } G_{20}. \\ \mathbb{Z}_{p^2} & \text{if } G \cong G_9. \\ \mathbb{Z}_2 & \text{if } G \cong G_{14} \text{ or } G_{15}. \\ \mathbb{Z}_3 & \text{if } G \cong G_{18}. \end{cases}$$

Now, we can obtain the tensor center of non-abelian groups of order p^4.

Theorem 2.8. *Let G be a group of order p^4, where p is prime. Then*

$$Z^{\otimes}(G) \cong \begin{cases} 1 & \text{if } G \cong G_6, G_{10}, G_{11}, G_{12}, G_{13}, G_{14}, G_{15} \text{ or } G_{19}. \\ \mathbb{Z}_p & \text{if } G \cong G_7, G_8, G_9, G_{16}, G_{17} \text{ or } G_{20}. \\ \mathbb{Z}_3 & \text{if } G \cong G_{18}. \end{cases}$$

And finally we obtain $\nabla(G)$ and $J_2(G)$ for non-abelian groups of order p^4 by following theorems

Theorem 2.9. *[2, Theorem 1.3] If G^{ab} has no elements of order 2, then $\nabla(G) \cong \Gamma(G^{ab})$.*

Theorem 2.10. *[2, Colollary 1.4] Let G be a group such that G^{ab} is a finitely generated abelian group with no elements of order 2. Then $J_2(G) \cong \Gamma(G^{ab}) \times \mathcal{M}(G)$.*

Lemma 2.11. *Let G be a non-abelian group of order p^4, where p is an odd prime. Then*

$$\nabla(G) \cong \begin{cases} \mathbb{Z}_p^{(6)} & \text{if } G \cong G_6, G_7 \text{ or } G_8. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)} & \text{if } G \cong G_9, G_{10}, G_{11}. \\ \mathbb{Z}_p^{(3)} & \text{if } G \cong G_{16}, G_{17}, G_{19}, G_{20}. \end{cases}$$

Lemma 2.12. *Let G be a non-abelian group of order p^4, where p is an odd prime. Then*

$$J_2(G) \cong \begin{cases} \mathbb{Z}_p^{(10)} & \text{if } G \cong G_6. \\ \mathbb{Z}_p^{(8)} & \text{if } G \cong G_7 \text{ or } G_8. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)} & \text{if } G \cong G_9. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(4)} & \text{if } G \cong G_{10}. \\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(3)} & \text{if } G \cong G_{11}. \\ \mathbb{Z}_p^{(4)} & \text{if } G \cong G_{16}, G_{17} \text{ or } G_{20}. \\ \mathbb{Z}_p^{(5)} & \text{if } G \cong G_{19}. \end{cases}$$

The following table contains results for non-abelian groups of order p^4 when $p > 3$.

TABLE 1. Fig. 1.

	Type of G	$\nabla(G)$	$G \wedge G$	$G \otimes G$	$Z^{\wedge}(G)$	$Z^{\otimes}(G)$	$J_2(G)$
$p > 2$	G_6	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$	1	1	$\mathbb{Z}_p^{(10)}$
$p > 2$	G_7	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(8)}$
$p > 2$	G_8	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(8)}$
$p > 2$	G_9	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$
$p > 2$	G_{10}	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p$	1	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(4)}$
$p > 2$	G_{11}	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(2)}$	1	1	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(3)}$
$p > 2$	G_{16}	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$
$p > 2$	G_{17}	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$
$p > 3$	G_{19}	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$	1	1	$\mathbb{Z}_p^{(5)}$
$p > 3$	G_{20}	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$

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On The Relative 2-Engel Degree Of Finite Groups

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Abstract

Let G be a finite group. The concept of n-Engel degree of G, denoted by $d_n(G)$, is the probability of two randomly chosen elements x and y of G satisfy the n-Engel condition $[y,_n x] = 1$. 1-Engel degree of the group G is the known commutativity degree of G. The aim of this paper is to define and investigate the relative 2-Engel degree of G and a subgroup H of G.

Keywords and phrases: relative commutativity degree, 2-Engel degree, 2-Engel group.

2010 *Mathematics subject classification:* Primary : 20F45; Secondary : 20F99.

1. Introduction

All groups which are considered in this paper are finite. For every group G the commutativity degree of G, denoted by $d(G)$, is defined as the probability that two randomly chosen elements of G commute, that is

$$d(G) = \frac{|\{(x, y) \in G \times G \mid [y, x] = 1\}|}{|G|^2}.$$

The commutativity degree of G was first introduced by P. Erdős and P. Turán in [4] and its generalizations are extensively studied in the literature and we may refer the reader to [1]. For a given natural number n, the n-Engel degree of G, denoted by $d_n(G)$, is defined as the probability that two randomly chosen elements x and y of G satisfy the n-Engel condition $[y,_n x] = 1$, that is

$$d_n(G) = \frac{|\{(x, y) \in G \times G \mid [y,_n x] = 1\}|}{|G|^2}.$$

* speaker

We have the following inequalites for the Engel degrees of a group G,

$$d_1(G) \leq d_2(G) \leq \cdots \leq d_n(G) \leq \cdots .$$

Recall that for a positive integer n, the notations $\pi(G)$, $L_n(G)$, $L(G)$, $R_n(G)$ and $R(G)$ denote the set of all prime divisors of the order of G, all left n-Engel elements of G, all left-Engel elements of G, all right n-Engel elements of G and all right-Engel elements of G, respectively.

Let χ be the class of all groups G such that $E_G(x) = \{y \in G : [y, x, x] = 1\}$ is a subgroup of G for all $x \in G$. For a given group G, note that $R_2(G)$ is always a subgroup of G, while $L_2(G)$ is not necessarily a subgroup. It is known that for an element $x \in G$, $E_G(x)$ is a subgroup of G whenever $[E_G(x), x, E_G(x), x] = 1$ or $[E_G(x), x]$ is abelian.

Erfanian, Rezaei and Lescot in [3], generalized the notation of $d(G)$ by defining the relative commutativity degree of G and a subgroup H, denoted by $d(H, G)$, which is the probability that an element of H commutes with an element of G. We generalize it to the notion of 2-Engel degree of G as follows:

Definition 1.1. *Let $g \in \chi$ and $H \leq G$. Then the relative 2-Engel degree of the subgroup H in the group G is defined as follows:*

$$d_2(H, G) = \frac{|\{(x, y) \in H \times G \mid [y, x, x] = 1\}|}{|H||G|}.$$

In this paper, we present some properties and results involving general lower and upper bounds for the relative 2-Engel degree of a group G and a subgroup H of G. We give some upper bounds for $d_2(H, G)$ when $G \in \chi$ and G is not a 2-Engel group.

2. Main Results

In this section, we have the main results on the relative 2-Engel degree of a subgroup H in a group G. We begin with the following lemma:

Lemma 2.1. *If $G \in \chi$ and $H \leq G$, then for all $x \in G$ we have $[H : E_H(x)] \leq [G : E_G(x)]$.*

In the next theorem, we compare the relative 2-Engel degree of a subgroup H in a group G with the 2-Engel degree of H:

Theorem 2.2. *If $H \leq G$, then $d_2(H, G) \leq d_2(H)$.*

In the following three theorems, we have some upper or lower bounds for the relative 2-Engel degree of a subgroup H in a group G:

Theorem 2.3. *Let $H \leq G$. Then*

$$d_2(H, G) \leq \frac{1}{2}(1 + \frac{|L_2(H)|}{|H|}).$$

Theorem 2.4. *Let $G \in \chi$ be a group which is not 2-Engel. If p is the smallest prime in $\pi(G)$, then*

$$d_2(H,G) \leq p([G:H]) + \frac{p-1}{p}(\frac{|L_2(G) \cap H|}{|H|});$$

In addition, if $L_2(H) \leq H$, then

$$d_2(H,G) \leq \frac{p^3+p-1}{p^2}([G:H]).$$

Theorem 2.5. *Let G be a nonabelian group and p be as the previous theorem. Then*

$$\frac{|L_2(G) \cap H|}{|H|} + p(\frac{|G| - |L_2(G) \cap H|}{|H||G|}) \leq d_2(H,G) \leq \frac{1}{2}([G:H] + \frac{|L_2(G) \cap H|}{|H|}).$$

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Regular Bipartite Divisor Graph

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Abstract

Given a finite group G, the *bipartite divisor graph*, denoted by $B(G)$, for its irreducible character degrees is the bipartite graph with bipartition consisting of $cd(G)^*$, where $cd(G)^*$ denotes the nonidentity irreducible character degrees of G and the $\rho(G)$ which is the set of prime numbers that divide these degrees, and with $\{p, n\}$ being an edge if $\gcd(p, n) \neq 1$. In this talk we consider the case that $B(G)$ is a regular graph and in particular, the case where $B(G)$ is a cycle.

Keywords and phrases: irreducible character degrees, regular graph .

2010 *Mathematics subject classification:* Primary 00Z99; secondary 99A00.

1. Introduction

Given a finite group G, it is an area of research to convey nontrivial information about the structure of G through some sets of invariants associated to G such as the set of degrees of the irreducible complex characters of G and would be interesting to distinguish the group structure of G influenced by these sets. In [1], the author considered the cases where $B(G)$ is a path or a cycle and discussed some properties of G. In particular she proved that $B(G)$ is a cycle if and only if G is solvable and $B(G)$ is either a cycle of length four or six. As cycles are special types of regular graphs, in this paper we consider the case where $B(G)$ is an n-regular graph for $n \in \{1, 2, 3\}$.

2. Main Results

It is well known that the set of irreducible characters of G, denoted by $Irr(G)$, can be used to obtain information about the structure of the group G. In this paper we are interested in the set of irreducible character degrees of G, that is, $cd(G) = \{\chi(1) : \chi \in Irr(G)\}$. When studying problems on character degrees, it is useful to attach the following graphs, which have been widely studied, to the sets $\rho(G)$ and $cd(G) \setminus \{1\}$.

(i) Prime degree graph, namely $\Delta(G)$, which is an undirected graph whose set of vertices is $\rho(G)$; there is an edge between two different vertices p and q if pq divides some degree in $cd(G)$.

(ii) Common divisor degree graph, namely $\Gamma(G)$, which is an undirected graph whose set of vertices is $cd(G) \setminus \{1\}$; there is an edge between two different vertices m and k if $gcd(m, k) \neq 1$.

Theorem 2.1. *Suppose G is a finite group and $B(G)$ is 1-regular. Then one of the following cases occurs:*

(i) If G is nonsolvable, then $n(B(G)) = 3$, $G \simeq A \times PSL(2, 2^n)$, where A is abelian and $n \in \{2, 3\}$.

(ii) If G is solvable, then for a prime p either $G \simeq P \times A$, where P is a nonabelian p-group and A is abelian, or G has an abelian normal subgroup of index a power of p.

Theorem 2.2. *Suppose that G is a finite group whose $B(G)$ is 2-regular. Then G is solvable and $B(G)$ is a cycle of length four or six.*

Theorem 2.3. *Let G be a solvable group whose $B(G)$ is a 3-regular graph. If $\Delta(G)$ is regular, then it is a complete graph. Furthermore, if $\Gamma(G)$ is not complete, then $\Delta(G)$ is isomorphic with K_n, for $n \geq 5$.*

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On finite groups whose self-centralizing subgroups are normal

M. Hassanzadeh* and Z. Mostaghim

Abstract

In this paper, we prove that in a finite group, every self-centralizing subgroup is normal iff it is a nilpotent group of class two. Also we show that finite groups, whose self-centralizing subgroups are subnormal are exactly nilpotent groups.

Keywords and phrases: self-centralizing subgroup, normal subgroup, maximal abelian subgroup, extraspecial p-group, nilpotent group.

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1. Introduction

Let G be a group and H be a subgroup of G. H is called a *self-centralizing subgroup* of G, if $C_G(H) \subseteq H$. It is equivalent to $C_G(H) = Z(H)$.

In an extraspecial p-group G, we have $Z(G) = G'$, so that every self-centralizing subgroup H is normal, since $G' = Z(G) \leq C_G(H) \leq H$.

Therefore a question naturally arises:

Question 1. *In which finite groups, all self-centralizing subgroups are normal?*

In Theorem 2.11, we prove that these groups are exactly nilpotent groups of class two.

Also a similar question can be proposed:

Question 2. *In which finite groups, all self-centralizing subgroups are subnormal?*

* speaker

In Theorem 2.12, we show that these groups are exactly nilpotent groups.

We present some preliminaries which are necessary.

Lemma 1.1. *(Upward-closedness) Let G be a group and $H \leq K \leq G$. If H is a self-centralizing subgroup in G, then K is a self-centralizing subgroup in G.*

Lemma 1.2. *The centralizer $C_G(A)$ is always self-centralizing in G, for any abelian subgroup $A \leq G$.*

Lemma 1.3. *[2,* Lemma 1*] The normalizer $N_G(H)$ is always self-centralizing in G, for any subgroup $H \leq G$.*

Lemma 1.4. *Maximal abelian subgroups of group G are self-centralizing.*

In this paper we consider finite groups.

2. Main Results

Here, we show that finite groups whose self-centralizing subgroups are normal, are exactly nilpotent groups of class two.

Definition 2.1. *We say that a group G is an ScN-group, if all self-centralizing subgroups of G are normal.*

Example 2.2. *Every nilpotent group of class two is an ScN-group. Particularly, extraspecial p-groups are ScN-groups.*

The following proposition is a well-known fact.

Proposition 2.3. *Let G be a finite group. Then every subgroup of G is subnormal iff G is nilpotent. ([4,* p.267, 11.3*])*

Proposition 2.4. *If G is a finite ScN-group, then G is nilpotent.*

Proof. By Lemma 1.3 and the last proposition. □

Now we recall a well-known result:

Proposition 2.5. *A Hamiltonian group is the direct product of a quaternion group with an abelian group in which every element is of finite odd order and an abelian group of exponent two. ([3,* p.190, Th. 12.5.4*])*

Proposition 2.6. *Let G be a finite ScN-group and N be a normal subgroup of G, then $\frac{G}{N}$ is an ScN-group.*

Also if N is a self-centralizing subgroup of G, then $\frac{G}{N}$ is an abelian or Hamiltonian group.

Corollary 2.7. *If G is a finite ScN-group of odd order and N is a self-centralizing subgroup of G, then $\frac{G}{N}$ is abelian.*

Proposition 2.8. *If G is a finite ScN-group of odd order, then $G' \leq Z(G)$.*

Proof. A maximal abelian subgroup $A \leq G$ is self-centralizing, hence $\frac{G}{A}$ is abelian. □

For the even order groups, it is enough to study 2-groups. We use the following theorem for this purpose.

Proposition 2.9 (Rocke). *Let G be a p-group in which the centralizer of each element is normal. Then either $cl(G) \leq 2$ or G is a 3-group of class 3. ([1, p.275, Th. 27.1])*

Proposition 2.10. *If G is a finite 2-group and ScN-group, then $G' \leq Z(G)$.*

Proof. The centralizer of each element is normal, thus by the last proposition, the assertion is proved. □

The main result of this paper, is the following theorem:

Theorem 2.11. *A finite group G is an ScN-group, i.e. all self-centralizing subgroups of G are normal, iff G is a nilpotent group of class two.*

In similar way, we can prove that:

Theorem 2.12. *All of self-centralizing subgroups of a finite group are subnormal iff it is nilpotent.*

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Almost Simple Groups and Their Non- Commuting Graph

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Abstract

Let G be a non-abelian finite group and $Z(G)$ be the center of G. The non-commuting graph, $\nabla(G)$ associated to G is the graph whose vertex set is $G - Z(G)$ and two distinct vertices x, y are adjacent if and only if $xy \neq yx$. We conjecture that if G is an almost simple group and H is a non-abelian finite group such that $\nabla(G) \cong \nabla(H)$, then $|G| = |H|$. Among other results, we prove that if $(G : S)$ is an almost simple group such that S is one of the Sporadic simple groups or S is one of the mentioned Lie Groups $L_2(7), L_2(8), L_2(17), L_3(3), U_3(3), U_4(2), F_4(2), O^+_{10}(2)$ and $O^-_{10}(2)$ and H is a non-abelian group with isomorphic non-commuting graphs, then $G \cong H$.

Keywords and phrases: Non-commuting graph; almost simple group; prime graph; OD-characterization; isomorphism.

2010 *Mathematics subject classification:* Primary: 22D15, 43A10.

1. Introduction

All groups under consideration are finite. For any group G, we denote $\pi(G)$ as the set of all prime divisors of G. A finite group G is called a simple K_n-group, if G is a simple group and $|\pi(G)| = n$. The prime graph $\Gamma(G)$ of a group G is a simple graph whose vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (written $p \sim q$) if and only if G contains an element of order pq. For $p \in \pi(G)$ we put $deg(p) = |\{q \in \pi(G) | p \sim q\}|$, which is called the degree of p. If $|G| = {p_1}^{\alpha_1} {p_2}^{\alpha_2} ... {p_k}^{\alpha_k}$, where p_i's are different primes, we define $D(G) = (deg(p_1), deg(p_2), ..., deg(p_k))$, which is called the degree pattern of G . The group G is called k -fold OD- characterizable if there exist exactly k nonisomorphic groups H satisfying conditions $|G| = |H|$ and $D(G) = D(H)$. A group G is called an almost simple group if $S \leq G \leq Aut(S)$ for some non-abelian simple group S and is denoted by $(G : S)$. Let G be a non-abelian group and $Z(G)$ be its center. We will associate a graph $\nabla(G)$ to G which is called the non-commuting graph of G. The vertex set of the non-commuting graph of G is $G - Z(G)$ and the edge set consists of $\{x, y\}$ where x and y are distinct non-central elements of G such that $xy \neq yx$. If G is a group, then $N(G) = \{n \in N | G$ has a conjugacy class C such that $|C| = n\}$. Recently in [1], some group and graph properties of the non-commuting graph associated to a non-abelian group are studied in particular, the authors put forward the following conjectures. Let G and H be two non-abelian finite groups such that $\nabla(G) \cong \nabla(H)$. Then $|G| = |H|$. Let S be a finite simple group and G be a group such that $\nabla(S) \cong \nabla(G)$. Then $G \cong S$. Recently the second conjecture has been proved by Solomon and Woldar, to see the details refer to [4]. Our aim in this paper is to prove the first conjecture for all the almost simple groups. We obtain that the certain almost simple groups with the same non-commuting graphs, are isomorphic together. we state some results which will be used in proving our main theorems. We denote $k(G)$ as the number of conjugacy classes of G.

Proposition 1.1. [3] *If G is a finite non-abelian group, then $k(G)/|G| \leq 5/8$.*

Proposition 1.2 (5). *Let p be a prime and q be a certain prime power. A finite group G is OD-characterizable if G is one of the following groups:*

1- The alternating groups A_p, A_{p+1} and A_{p+2};

2- All finite almost simple K_3 -groups except $Aut(A_6)$ and $Aut(U_4(2))$;

3- The symmetric groups S_p and $S_p + 1$;

4- All finite simple group K_4 -groups except A_{10};

5- The simple groups of Lie type $L_2(q), L_3(q), U_3(q),^2 B_2(q)$ and $^2G_2(q)$.

6- All sporadic simple groups and their automorphism groups except $Aut(J_2)$ and $Aut(MCL)$.

Proposition 1.3 (2). *Let G be a group with $Z(G) = 1$ and let M be one of $Aut(J_2)$ and $Aut(MCL)$ satisfying $N(G) = N(M)$. Then $G \cong M$.*

2. Main Results

In this section, we present our main results.

Theorem 1. *Let $(G : S)$ be an almost simple group. If H is a group such that $\nabla(G) \cong \nabla(H)$, then $|G| = |H|$.*

Proposition 2.1. *Let $(G : S)$ be an almost simple group and H be a group such that $\nabla(G) \cong \nabla(H)$. Then $D(G) = D(H)$.*

Proposition 2.2. *Let $(G : S)$ be an almost simple group and H be a group such that $\nabla(G) \cong \nabla(H)$. Then $N(G) = N(H)$.*

Theorem 2. *Let $(G : S)$ be an almost simple group such that S is one of the Sporadic overflow simple groups or S is one of the mentioned Lie $L_2(7), L_2(8), L_2(17), L_3(3), U_3(3), U_4(2), F_4(2), O^+_{10}(2)$ and $O^-_{10}(2)$. If H is a group such that $\nabla(G) \cong \nabla(H)$, then $G \cong H$.*

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The Schur multiplier, tensor square and capability of free nilpotent Lie algebras

Farangis Johari*, Peyman Niroomand and Mohsen Parvizi

Abstract

In this talk, we determine the structure of the tensor square and Schur multiplier of free nilpotent Lie algebras. Among the other results, we show all such finite dimensional Lie algebras are capable.

Keywords and phrases: Free nilpotent Lie algebras, Tensor square, Exterior square, Schur multiplier, Capable Lie algebras.

2010 *Mathematics subject classification:* Primary: 17B30; Secondary: 17B05,17B99.

1. Introduction and Motivation

All Lie algebras are considered over a fixed field F and $[,]$ denotes the Lie bracket.

A bilinear function $\alpha : L \times L \to T$ is an exterior Lie pairing if the following relations are satisfied:

$$\alpha([x, x'], y) = \alpha(x, [x', y]) - \alpha(x', [x, y]),$$

$$\alpha(x, [y, y']) = \alpha([y', x], y) - \alpha([y, x], y'),$$

$$\alpha([y, x], [x', y']) = -[\alpha(x, y), \alpha(x', y')],$$

for all $x, x', y, y' \in L$.

Definition 1.1. *Let L be a Lie algebra over a fixed field F. The non-abelian tensor square $L \otimes L$ of L is the Lie algebra generated by the symbols $l \otimes x$ with the following defining relations:*

$$c(l \otimes x) = cl \otimes x = l \otimes cx, (l + l') \otimes x = l \otimes x + l' \otimes x,$$

* speaker

$$l \otimes (x + x') = l \otimes x + l \otimes x', \ [l, l'] \otimes x = l \otimes \ [l', x] - l' \otimes \ [l, x],$$

$$l \otimes \ [x, x'] = \ [x', l] \otimes x - \ [x, l] \otimes x', [l \otimes x, l' \otimes x'] = - [x, l] \otimes \ [l', x'],$$

for all $c \in F$, $l, l', x, x' \in L$.

Let $L\square L$ denotes the submodule of $L \otimes L$ generated by $l \otimes l$, for all $l \in L$. Then the exterior square $L \wedge L$ of L is the quotient $(L \otimes L)/L\square L$. For all $l \otimes l \in L \otimes L$, we denote the coset $(l \otimes l) + L\square L$ by $l \wedge l$.

Let L be a Lie algebra presented as the quotient of a free Lie algebra F by an ideal R. Then the Schur multiplier of L is defined to be $\mathcal{M}(L) \cong (R \cap F^2)/[R, F]$, where F^2 is the derived subalgebra of F. (See also [1] for more information on the Schur multiplier of Lie algebras). It is obvious that $\mathcal{M}(L)$ is abelian and independent of the choice of the free Lie algebra F.

Also from [2], the kernel of the commutator map $\kappa' : L \wedge L \to L^2$ given by $l \wedge l \mapsto [l, l]$ is isomorphic to the Schur multiplier of L.

Let $(c + 1)$-th term of the lower central series of L is denoted by L^{c+1}, where $L^1 = L$ and $L^{c+1} = [L^c, L]$, inductively. Then

Definition 1.2. *A Lie algebra L is called a free nilpotent Lie algebra of rank $n > 1$ and class $i - 1$ provided that $L \cong F/F^i$, where F is a free Lie algebra of rank $n > 1$. It is denoted by $\mathcal{F}_i$.*

In fact, $\mathcal{F}_i$ is a free object in the variety of the nilpotent Lie algebra of nilpotency class at most i. In this paper, we are going to explicit the structures of Schur multiplier and the tensor square of finite dimensional free nilpotent Lie algebras. Then we prove that they are also capable.

2. Main Results

In this section, we are going to determine the Schur multiplier of a free nilpotent Lie algebra. Then we describe the tensor and exterior square of finite dimensional free nilpotent Lie algebras. Finally, we show that all finite-dimensional free nilpotent Lie algebras are capable.

Recall that a pair of Lie algebras (K, M) is said to be a defining pair for L if $M \subseteq Z(K) \cap K^2$ and $K/M \cong L$. If L is finite-dimensional, then the dimension of K is bounded. If (K, M) is a defining pair for L, then the first component of maximal dimension is called a cover for L. Moreover, from [1], in this case $M \cong \mathcal{M}(L)$.

Proposition 2.1. *Let F be a free Lie algebra. Then $\mathcal{F}_{i+1}$ is a cover of $\mathcal{F}_i$ and $\mathcal{M}(F_i) \cong \mathcal{F}^i_{i+1}$, for $i \geq 2$.*

Proof. Clearly, $\mathcal{M}(F_i) \cong F^i/F^{i+1} \cong \mathcal{F}^i_{i+1}$. Since $\mathcal{F}^i_{i+1} \subsetneqq Z(\mathcal{F}_{i+1}) \cap \mathcal{F}^2_{i+1}$, we have $\mathcal{F}_{i+1}/\mathcal{F}^i_{i+1} \cong \mathcal{F}_i$. The result follows. □

Theorem 2.2. *[4, Theorem 2.10] Let* $0 \to R \to F \to L \to 0$ *be a free presentation of a Lie algebra* L*. Then* $L \wedge L \cong F^2/[R, F]$.

There exists a relation between the exterior square of $\mathcal{F}_i$ and the derived subalgebra of $\mathcal{F}_{i+1}$ as follows

Theorem 2.3. *Let* L *be a free nilpotent Lie algebra. Then* $L \wedge L \cong \mathcal{F}_{i+1}^2$*, for* $i \geq 2$.

Proof. By Theorem 2.2, we have $L \wedge L \cong F^2/[F^i, F] \cong \mathcal{F}_{i+1}^2$, as required. □

The notion of WhiteheadâĂŹs quadratic functor Γ is defined in [2] for Lie algebras. Let $\psi : \Gamma(L/L^2) \to L \wedge L$ given by $(\gamma(l + L^2) \mapsto l \otimes l$, be the natural homomorphism. Then

Lemma 2.4. *Let* F *be a free Lie algebra. Then the sequence* $0 \to \Gamma(F^{ab}) \to \mathcal{F}_i \otimes \mathcal{F}_i \to \mathcal{F}_i \wedge \mathcal{F}_i \to 0$ *is exact, for* $i \geq 2$.

Proof. The result follows from [2, Proposition 17]. □

Corollary 2.5. *Let* F *be a free Lie algebra. Then the sequence* $0 \to \Gamma(F^{ab}) \to \mathcal{F}_i \otimes \mathcal{F}_i \to \mathcal{F}_{i+1}^2 \to 0$ *is exact. Moreover,* $F \square F \cong \mathcal{F}_i \square \mathcal{F}_i$*, for* $i \geq 2$.

Proof. By Theorem 2.3 and Lemma 2.4, we have the exact sequence $0 \to \Gamma(F^{ab}) \to \mathcal{F}_i \otimes \mathcal{F}_i \to \mathcal{F}_{i+1}^2 \to 0$. By using the epimorphism ψ and [2, Proposition 17], we get $F \square F \cong \Gamma(F^{ab}) \cong \mathcal{F}_i \square \mathcal{F}_i$. The proof is completed. □

Following Shirshov [7], for a free Lie algebra L on the set $X = \{x_1, x_2, \ldots\}$, the basic commutators on the set X defined inductively as follows.

(i) The generators $x_1, x_2, \ldots, x_n$ are basic commutators of length one and ordered by setting $x_i < x_j$ if $i < j$.

(ii) If all the basic commutators d_i of length less than t have been defined and ordered, then we may define the basic commutators of length t to be all commutators of the form $[d_i, d_j]$ such that the sum of lengths of d_i and d_j is t, $d_i > d_j$, and if $d_i = [d_s, d_t]$, then $d_j \geq d_t$. The basic commutators of length t follow those of lengths less than t. The basic commutators of the same length can be ordered in any way, but usually the lexicographical order is used.

The number of all basic commutators on a set $X = \{x_1, x_2, \ldots x_d\}$ of length n is denoted by $l_d(n)$. Thanks to [7], we have

$$l_d(n) = \frac{1}{n} \sum_{m|n} \mu(m) d^{nm},$$

where $\mu(m)$ is the Möbius function, defined by $\mu(1) = 1, \mu(k) = 0$ if k is divisible by a square, and $\mu(p_1 \dots p_s) = (-1)^s$ if $p_1, \dots, p_s$ are distinct prime numbers. Using the topside statement and looking [7], we have the next theorem.

Theorem 2.6. *Let F be a free Lie algebra on a set $X = \{x_1, x_2, \dots x_d\}$. Then F^c/F^{c+i} is an abelian Lie algebra with the basis of all basic commutators on X of lengths $c, c+1, \dots, c+i-1$, for all $0 \leq i \leq c$. In particular, F^c/F^{c+1} is an abelian Lie algebra of dimension $l_d(c)$.*

We denote an abelian Lie algebra of dimension n by $A(n)$. In the following result, we compute the dimension of Schur multiplier of a finite dimensional free nilpotent Lie algebra.

Theorem 2.7. *Let F be a free Lie algebra on a set $X = \{x_1, x_2, \dots x_d\}$. Then $\mathcal{M}(\mathcal{F}_i) \cong A(l_d(i))$ and $F \square F \cong \mathcal{F}_i \square \mathcal{F}_i \cong A(l_{d+1}(2))$.*

Proof. By Proposition 2.1 and Theorem 2.6, we have $\mathcal{M}(\mathcal{F}_i) \cong A(l_d(i))$. Corollary 2.5 and [4, Lemma 2.3] imply $F \square F \cong \mathcal{F}_i \square \mathcal{F}_i \cong A(l_{d+1}(2))$, as required. □

Corollary 2.8. *Let F be a free Lie algebra on a set $X = \{x_1, x_2, \dots x_d\}$. Then $J_2(\mathcal{F}_i) \cong A(l_{d+1}(2) + l_d(i))$, for $i \geq 2$.*

Proof. The result follows from [4, Corollary 2.6] and Theorem 2.7. □

The following theorem determines the structure of tensor square of a finite dimensional free nilpotent Lie algebra.

Theorem 2.9. *Let F be a free Lie algebra on a set $X = \{x_1, x_2, \dots x_d\}$. Then $\mathcal{F}_i \otimes \mathcal{F}_i \cong \mathcal{F}_{i+1}^2 \oplus A(l_{d+1}(2))$, for $i \geq 2$.*

Proof. The result follows from [4, Theorem 2.5], Theorems 2.3 and 2.7. □

Recall from [6] that a Lie algebra is capable provided that $L \cong H/Z(H)$ for a Lie algebra H. In [6], the epicenter of a Lie algebra L, $Z^*(L)$, is defined to be the smallest ideal M of L such that L/M is capable and it is shown that L is capable if and only if $Z^*(L) = 0$.

Lemma 2.10. *Let F be a free Lie algebra on a set $X = \{x_1, x_2, \dots x_d\}$. Then $Z^*(\mathcal{F}_i) \subsetneqq \mathcal{F}_i^2$, for $i \geq 2$.*

Proof. By [5, Theorem 3.3], we have $\mathcal{F}_i/\mathcal{F}_i^2$ is capable. Thus $Z^*(\mathcal{F}_i) \subsetneqq \mathcal{F}_i^2$, as required. □

Our approach is based on the concept of the exterior center $Z^\wedge(L)$, the set of all elements l of L for which $l \wedge l' = 0_{L \wedge L}$ for all $l' \in L$. Niroomand et al. in [5] showed $Z^\wedge(L) = Z^*(L)$ for any finite dimensional Lie algebra L. Recently, the authors in [3] proved $Z^{\wedge}(L) = Z^{*}(L)$ for any Lie algebra L. [3, Corollary 2.3.18]. It allows to say that L is capable if and only if $Z^{\wedge}(L) = 0$.

Corollary 2.11. *Let F be a free Lie algebra on a set $X = \{x_1, x_2, \ldots x_d\}$. Then $Z^{\otimes}(\mathcal{F}_i) = Z^{\wedge}(\mathcal{F}_i)$, for $i \geq 2$.*

Proof. Lemma 2.10 and [4, Corollary 2.7] imply $Z^{\otimes}(\mathcal{F}_i) = Z^{\wedge}(\mathcal{F}_i)$. The result holds. □

Theorem 2.12. *Let F be a free Lie algebra on a set $X = \{x_1, x_2, \ldots x_d\}$. Then $Z(\mathcal{F}_i) = \mathcal{F}_i^{i-1}$, for $i \geq 2$.*

Proof. We may prove the result by induction on i. If $i = 2$, then the result holds. Let $i \geq 3$. By induction hypothesis, we have $Z(\mathcal{F}_i) = \mathcal{F}_i^{i-1}$. Consider the natural epimorphism $\psi : \mathcal{F}_{i+1} \to \mathcal{F}_i$. We conclude that $\mathcal{F}_{i+1}^{i} = F^i/F^{i+1} \subseteq Z(\mathcal{F}_{i+1}) \subseteq F^{i-1}/F^{i+1}$. Let $x \in Z(\mathcal{F}_{i+1})$. The generators $x_1, x_2, \ldots, x_d$ are basic commutators of length one and ordered by setting $x_i < x_j$ if $i < j$. By Theorem 2.6, $x = a + b$, $a = \sum_{t=1}^{k} \alpha_t y_t$ and $b = \sum_{j=1}^{m} \beta_j b_j$ such that y_t is a basic commutator of weight i, b_j is a basic commutator of weight $i - 1$ and α_t and β_j are scaler. We have $0 = [x, x_d] = [a, x_d] + [b, x_d] = \sum_{j=1}^{m} \beta_j [b_j, x_d]$. Since clearly $[b_j, x_d]$ is a basic commutator of weight i for all $1 \leq j \leq m$ and independent, we obtain $\beta_j = 0$ for all $1 \leq j \leq m$. Thus $x \in \mathcal{F}_i^{i}$ and so $Z(\mathcal{F}_{i+1}) = \mathcal{F}_i^{i}$. The result holds.

Theorem 2.13. *Let F be a free Lie algebra on set $X = \{x_1, x_2, \ldots x_d\}$. Then $\mathcal{F}_i$ is capable, for $i \geq 2$.*

Proof. We claim that $\mathcal{F}_i$ is capable, for $i \geq 2$. By using Theorem 2.12, we have $\mathcal{F}_i \cong \mathcal{F}_{i+1}/\mathcal{F}_{i+1}^{i} = \mathcal{F}_{i+1}/Z(\mathcal{F}_{i+1})$. The result follows. □

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On isoclinism between pairs of n-Lie algebras

Azam K. Mousavi

Abstract

In the present paper we study the notion of isoclinism on a pair of n-Lie algebras, which forms an equivalence relation and show that each equivalence class contains a stem pair of n-Lie algebras, which has minimal dimension amongst the finite dimensional pairs of n-Lie algebras. Finally, some more results are obtained when two isoclinic pairs of n-Lie algebras are given.

Keywords and phrases: n-Lie Algebra, Isoclinism.

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1. Introduction

In 1987, Filippov [2] introduced the notion of n-Lie algebras and classified all n-Lie algebras of dimension $n + 1$ over an algebraically closed field.

An n-Lie algebra is a vector space L over a field F together with the following n-linear map

$$[-, \dots, -] : L \times \cdots \times L \longrightarrow L$$

given by

$$(x_1, \dots, x_n) \longmapsto [x_1, \dots, x_n],$$

for all $x_i \in L$ such that the following conditions hold:

(i) $[x_1, \dots, x_i, \dots, x_j, \dots, x_n] = 0$, when $x_i = x_j$; and

(ii) $[[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^{n} [x_1, \dots, [x_i, y_2, \dots, y_n], \dots, x_n]$,

for all $x_i, x_j \in L$, $1 \le i \le n$ and $2 \le j \le n$.

A subspace S of n-Lie algebra L, which is closed under the n-Lie product is said to be *n-Lie subalgebra* of L. Also, n-Lie subalgebra I of L is called *n-Lie ideal* of L if $[I, \underbrace{L, \ldots, L}_{(n-1)-\text{times}}] \subseteq I$.

The notion of isoclinism for Lie algebras is introduced, by Moneyhun [4] in 1994 and it is extended by Moghaddam and Parvaneh in recent years [3, 5]. Eshrati, Moghaddam and Saeedi [1] generalized the notion of isoclinism to n-Lie algebras in 2016. They proved that the notion of isoclinism and isomorphism are equivalent for any two n-Lie algebras of the same dimensions.

Definition 1.1. *Let M be an ideal of an n-Lie algebra L. Then (M, L) is considered to be a pair of n-Lie algebra and one may define the commutator and the centre of the pair (M, L) as follows:*

$$[M, L, \ldots, L] = \langle [m, l, \ldots, l] : m \in M, l \in L \rangle$$

and

$$Z(M, L) = \{m \in M : [m, l, \ldots, l] = 0, \forall l \in L\}.$$

In this paper, we introduce the notion of isoclinism for the pairs of n-Lie algebras. Also, by using the notion of stem pair of n-Lie algebras, we shall prove the following: Any two stem pairs of n-Lie algebras (M, L) and (J, K), are isomorphic if and only if they are isoclinic (*See* Theorem 3.8).

Definition 1.2. *Two pairs of n-Lie algebras (M_1, L_1) and (M_2, L_2) are called isoclinic, denoted by $(M_1, L_1) \sim (M_2, L_2)$, if there exist a pair of isomorphisms (α, β), in which $\alpha : L_1/Z(M_1, L_1) \longrightarrow L_2/Z(M_2, L_2)$ with $\alpha(\overline{M}_1) = \overline{M}_2$, $\overline{M}_j = \frac{M_j}{Z(M_j, L_j)}$, $j = 1, 2$, and $\beta : [M_1, L_1, \ldots, L_1] \longrightarrow [M_2, L_2, \ldots, L_2]$ satisfies $\beta([m_1, l_2, \ldots, l_n]) = [m_2, l'_2, \ldots, l'_n]$, for all $m_1 \in M_1, l_i \in L_1, m_2 \in \alpha(\overline{m}_1)$, $l'_i \in \alpha(\bar{l}_i)$ and $2 \leq i \leq n$, and that the following diagram is commutative:*

$$\begin{array}{ccccc} \frac{M_1}{Z(M_1,L_1)} \times & \frac{L_1}{Z(M_1,L_1)} \times \cdots \times \frac{L_1}{Z(M_1,L_1)} & \longrightarrow & [M_1, L_1, \ldots, L_1] \\ \downarrow \alpha & \downarrow \alpha^{n-1} & & \downarrow \beta \\ \frac{M_2}{Z(M_2,L_2)} \times & \frac{L_2}{Z(M_2,L_2)} \times \cdots \times \frac{L_2}{Z(M_2,L_2)} & \longrightarrow & [M_2, L_2, \ldots, L_2], \end{array}$$

The horizontal maps are defined naturally.

2. Some Properties of Pairs of n-Lie Algebras

This section is devoted to showing some properties of a pair of n-Lie algebras. Particularly, a stem pair of n-Lie algebras will be introduced. It is also shown that any family of isoclinism pairs of n-Lie algebras contains a stem pair of n-Lie algebras.

Definition 2.1. *The pair (N, K) of n-Lie algebras is called abelian if $Z(N, K) = N$ or equivalently $[N, K, \dots, K] = 0$.*

Remark 2.2. *It is known that for any n-Lie algebra L, there exists an isoclinism $L \sim L \oplus A$, for any abelian n-Lie algebra A. However, this result does not hold for the pairs of n-Lie algebras. If (M, L) is a pair of n-Lie algebras and (N, K) is an abelian pair of n-Lie algebras, then (M, L) is not isoclinic to $(M \oplus N, L \oplus K)$, because*

$$Z(M \oplus N, L \oplus K) = Z(M, L) \oplus N$$

so that

$$\frac{L \oplus K}{Z(M \oplus N, L \oplus K)} \cong \frac{L \oplus K}{Z(M, L) \oplus N} \neq \frac{L}{Z(M, L)}.$$

To see this, we note that in the case of finite dimensions,

$$\dim \frac{L \oplus K}{Z(M \oplus N, L \oplus K)} = (\dim L - \dim Z(M, L)) + (\dim K - \dim N) \neq \dim \frac{L}{Z(M, L)}.$$

Theorem 2.3. *Let (M, L) be a pair of n-Lie algebras, K a subalgebra of L and $N \subseteq M$ be an ideal of L. Then*

(i) $(K \cap M, K) \sim (K \cap M + Z(M, L), K + Z(M, L))$. In particular, if $L = K + Z(M, L)$, then $(K \cap M, K) \sim (M, L)$. Conversely, if $K/Z(K \cap M, K)$ satisfies the descending chain condition on ideals and $(K \cap M, K) \sim (M, L)$, then $L = K + Z(M, L)$.

(ii) $(M/N, L/N) \sim (M/N \cap [N, L, \dots, L], L/N \cap [N, L, \dots, L])$. In particular, if $N \cap [N, L, \dots, L] = 0$, then $(M/N, L/N) \sim (M, L)$. Conversely, if $[M, L, \dots, L]$ satisfies the increasing chain condition on ideals and that $(M/N, L/N) \sim (M, L)$, then $N \cap [N, L, \dots, L] = 0$.

Definition 2.4. *The pair (M, L) of n-Lie algebras is called a stem pair of n-Lie algebras if $Z(M, L) \subseteq [M, L, \dots, L]$.*

In what follows, we shall prove that in every isoclinism family of pairs of n-Lie algebras, there always exists a stem pair, say (N, K), in such a way that K has minimum dimension.

Theorem 2.5. *Let C be an isoclinism family of pairs of n-Lie algebras. Then*

(i) C contains at least one stem pair of n-Lie algebras.

(ii) If (N, K) belongs to C, in which K has finite dimension, then (N, K) is a stem pair of n-Lie algebras if and only if $\dim K = \min\{\dim L | (M, L) \in C\}$.

Theorem 2.6. *Let (N_1, K_1) and (N_2, K_2) be isoclinic stem pairs of n-Lie algebras. Then $Z(N_1, K_1) \cong Z(N_2, K_2)$.*

3. Some Properties of the Factor Set on a Pair of n-Lie Algebras

In this section, we introduce factor sets on pairs of n-Lie algebras and apply to obtain some results on stem pairs of n-Lie algebras of finite dimension.

Definition 3.1. *Let (M, L) be a pair of n-Lie algebras and put $\overline{M} = M/Z(M, L)$. A factor set on the pair of n-Lie algebras (M, L) is defined as an n-linear map*

$$f : \frac{M}{Z(M,L)} \times \frac{M}{Z(M,L)} \times \cdots \times \frac{M}{Z(M,L)} \longrightarrow Z(M,L),$$

such that for all $x_1, \ldots, x_n, y_2, \ldots, y_n \in M$, the following conditions are satisfied:

(1) $f(\overline{x}_1, \ldots, \overline{x}_i, \ldots, \overline{x}_j, \ldots, \overline{x}_n) = 0$ *if and only if* $\overline{x}_i = \overline{x}_j$;

(2) $f([\overline{x}_1, \ldots, \overline{x}_n], \overline{y}_2, \ldots, \overline{y}_n) = f([\overline{x}_1, \overline{y}_2, \ldots, \overline{y}_n], \overline{x}_2, \ldots, \overline{x}_n)$
$+ f(\overline{x}_1, [\overline{x}_2, \overline{y}_2, \ldots, \overline{y}_n], \overline{x}_3, \ldots, \overline{x}_n)$
$+ f(\overline{x}_1, \overline{x}_2, [\overline{x}_3, \overline{y}_2, \ldots, \overline{y}_n], \overline{x}_4, \ldots, \overline{x}_n)$
$+ \cdots + f(\overline{x}_1, \ldots, \overline{x}_{n-1}, [\overline{x}_n, \overline{y}_2, \ldots, \overline{y}_n])$
$= \sum_{i=1}^{n} f(\overline{x}_1, \ldots, \overline{x}_{i-1}, [\overline{x}_i, \overline{y}_2, \ldots, \overline{y}_n], \overline{x}_{i+1}, \ldots, \overline{x}_n)$.

Taking the above notation, we have the following

Lemma 3.2. *Let f be a factor set on the pair of n-Lie algebras (M, L). Then*

(i) $M_f = Z(M, L) \times \overline{M} = \{(z_i, \overline{m}_i) : z_i \in Z(M, L), \overline{m}_i \in \overline{M}, i = 1, \ldots, n\}$ *is a stem n-Lie algebra under the component-wise addition, and the multiplication defined as follows:*

$$[(z_1, \overline{m}_1), \ldots, (z_n, \overline{m}_n)] = (f(\overline{m}_1, \ldots, \overline{m}_n), [\overline{m}_1, \ldots, \overline{m}_n]),$$

for all $z_i \in Z(M, L)$ and $\overline{m}_i \in \overline{M}$ $(i = 1, \ldots, n)$.

(ii) $Z_{M_f} = \{(z, 0) \in M_f : z \in Z(M, L)\} \cong Z(M, L)$.

The following lemma is needed to prove our next result.

Lemma 3.3. *Let (M, L) be a pair of n-Lie algebras. Then there exists a factor set f on (M, L) such that $M \cong Z(M, L) \times \overline{M} = M_f$.*

The following result establishes a relationship between stem pairs and factor sets in n-Lie algebras.

Theorem 3.4. *Let C be a family of isoclinic pairs of n-Lie algebras and (M, L) be a stem pair in C. Then for every stem pair (J, K) of C, there exists a factor set f on (M, L) satisfying*

$$J \cong Z(J, K) \times \overline{J} = J_g \cong M_f = Z(M, L) \times \overline{M} \cong M.$$

Lemma 3.5. *Let f and g be two factor sets on the pair of n-Lie algebras (M, L). If $h : M_f \longrightarrow M_g$ is an isomorphism satisfying $h(Z_{M_f}) = Z_{M_g}$, then h induces the automorphisms h_1 and h_2 on n-Lie algebras $\overline{M}$ and $Z(M, L)$, respectively.*

Theorem 3.6. *The isomorphism $h : M_f \longrightarrow M_g$ satisfying $h(Z_{M_f}) = Z_{M_g}$ induces the automorphisms h_1 of $M/Z(M,L)$ and h_2 of $Z(M,L)$ if and only if there exists a linear map $\gamma : M/Z(M,L) \longrightarrow Z(M,L)$ such that*

$$h_2(g(\overline{m}_1,\ldots,\overline{m}_n)+\gamma[\overline{m}_1,\ldots,\overline{m}_n]) = g(h_1(\overline{m}_1),\ldots,h_1(\overline{m}_n)),$$

for all $\overline{m}_1,\ldots,\overline{m}_n \in M/Z(M,L)$.

Definition 3.7. *The pairs of n-Lie algebras (M,L) and (J,K) are isomorphic if $M \cong J$ and $L \cong K$.*

In the following, we shall prove our main results of this section.

Theorem 3.8. *Let (M,L) and (J,K) be stem pairs of n-Lie algebras of finite dimension. Then $(M,L) \sim (J,K)$ if and only of $(M,L) \cong (J,K)$.*

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A note on the non-abelian tensor square of p-gorups

Elaheh Khamseh

Abstract

Let G be a non-abelian d-generator finite p-group of order p^n with $|G^{'}| = p^k$. In 1991, Rocco prove that $|G \otimes G| \leq p^{n(n-k)}$ that depends on n, k. In 2016, Jafari proved that $|G \otimes G| \leq p^{(n-1)d+2}$ that depends on n, d. In this paper, we obtain a new upper bound in terms of n, k, and d. In fact, we prove $|G \otimes G| \leq p^{(n-k)^2+d(n+k-2)+4}$, for $p \neq 2$ and find the structure of all p-groups that attains the mentioned bound.

Keywords and phrases: Non-abelian tensor square, Schur multiplier .

2010 *Mathematics subject classification:* Primary: 20D15; Secondary: 20J06.

1. Introduction

Let G and H be groups which act on themselves by conjugation and each of which acts on the other with an action $(g, h) \mapsto\ {}^{g}h$ of G on H and an action $(h, g) \mapsto\ {}^{h}g$ of H on G, in such a way that for all $g,\ g^{'} \in G$ and $h,\ h^{'} \in H$ the following compatibility holds:

$$({}^{g}h)g^{'} = {}^{ghg^{-1}}g^{'}, \qquad ({}^{h}g)h^{'} = {}^{hgh^{-1}}h^{'},$$

where ghg^{-1}, hgh^{-1} are here interpreted as elements of the free product of G and H. The non-abelian tensor product $G \otimes H$ is the group generated by all symbols $g \otimes h$, subject to the relations

$$gg^{'} \otimes h = ({}^{g}g^{'} \otimes {}^{g}h)(g \otimes h), \quad g \otimes hh^{'} = (g \otimes h)({}^{h}g \otimes {}^{h}h^{'}),$$

for all $g,\ g^{'} \in G$ and $h,\ h^{'} \in H$ that was introduced by Brown and Loday in 1987 [2]. This concept is closely related to other important subjects such as Schur multiplier, capability and topological algebra.

Using the conjugation action of a group on itself, one may always define the non-abelain tensor square $G \otimes G$. Ellis [3] proved that if G is a finite group, then so is $G \otimes G$. The structure of $G \otimes G$ and hence its order is computed, for some finte groups. In 1991, Rocco [7] proved that if G is a finite d-generator group of order p^n with $|G'| = p^k$, then

$$p^{d^2} \leq |G \otimes G| \leq p^{n(n-k)}.$$

Also Jafari [4] in 2016 showed that $|G \otimes G| \leq p^{(n-1)d+2}$. In this paper we use the relation between Schur multiplier of G and tensor square to obtain a new upper bound for $|G \otimes G|$. Also we find all p-groups which attain the bound are obtained.

2. Main Results

The Schur multiplier, $M(G)$, of a group G was appeared in works of Schur in 1904 during results of the projective representation of groups. There are some upper bounds for Schur multiplier of finite p-group G. In 2017, Rai [6] proved the following result is stronger that the other bounds.

Lemma 2.1. *[6] Let G be a non-abelian p-group of order p^n with $|G'| = p^k$ and $d(G) = d$, then*

$$|M(G)| \leq p^{\frac{1}{2}(d-1)(n+k-2)+1}.$$

In the following results, the order of the non-abelain tensor square of a finite group G is expressed in terms of the orders of G and Schur multiplier of G and $G^{ab} = \dfrac{G}{G'}$.

Lemma 2.2. *[1] Let G be a finite p-group. If $p > 2$, then $|G \otimes G| = |G||M(G)||M(G^{ab})|$.*

We prove the following theorem using lemmas 2.1, 2.2 and the results of Niroomand et al. [5].

Theorem 2.3. *Let G be a non-abelian d-generator finite p-group of order p^n with $|G'| = p^k$, then*

$$|G \otimes G| \leq p^{(n-k)^2+d(n+k-2)+4}$$

for $p \neq 2$, the bound is attained if and only if G is isomorphic to one of the following groups:
(i) $G_1 \simeq E_1 \times Z_p^{(n-3)}$, where E_1 is the extra-special p-group of order p^3 and exponent p, $(p \neq 2)$.
(ii) $G_2 \simeq \langle a, b \mid a^{p^m} = b^{p^m} = c^p = 1, [a,b] = c, [a,b,a] = [a,b,b] = 1\rangle$ is non-metacyclic of order p^{2m+1} and $m > 1$, $p \neq 2$.
(iii) $G_3 \simeq Z_p \rtimes Z_p^{(4)}$, $(p \neq 2)$.
(iv) $G_4 \simeq \langle a, b \mid a^{p^m} = b^{p^m} = c^{p^m} = 1, [a,b] = c, [a,b,a] = [a,b,b] = 1\rangle$ of order p^{3m}, where $2 \leq m$ and $(p \neq 2)$.

(v) $G_5 \simeq \langle x_1, y_1, x_2, y_2, x_3, y_3 \mid [x_1, x_2] = y_3, [x_2, x_3] = y_1, [x_3, x_1] = y_2, [x_i, y_j] = 1, x_i^p = y_i^p = 1, 1 \leq i, j \leq 3\rangle$ *and* $p \neq 2$.

(vi) $G_6 \simeq \langle x_1, y_1, x_2, y_2, x_3, y_3, z \mid [x_1, x_2] = y_3, [x_2, x_3] = y_1, [x_3, x_1] = y_2, [y_i, x_i] = z, [x_i, y_j] = 1, x_i^3 = y_i^3 = z^3 = 1, 1 \leq i, j \leq 3\rangle$.

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Centralizers and norm of a group

K. Khoramshahi* and M. Zarrin

Abstract

Let G be a group and $C(G)$ denote the intersection of the normalizers of centralizers of all elements of G. Put $C_0(G) = 1$ and define $C_{i+1}(G)/C_i(G) = C(G/C_i(G))$ for $i \geq 0$. Denote by $C_\infty(G)$ the terminal term of this ascending series. First we show, among other things, some basic properties of $C_n(G)$ and $C_\infty(G)$ and then give a new characterization for nilpotent groups in terms of series in which defined via $C_n(G)$.

Keywords and phrases: Norm, Centralizers, Baer groups, Engel groups .

2010 *Mathematics subject classification:* Primary: 20E34; Secondary: 20F45.

1. Introduction

For any group G, the norm (or Kern), denoted by $K(G)$, of G is the intersection of all the normalizers of subgroups of G. In particular, $K(G)$ is the intersection of all the normalizers of the non-normal subgroups of G with the observation that $K(G) = G$ if G is a Dedekind group; i.e., if G has no non-normal subgroups. This subgroup was introduced in 1935 by Baer [1] who delineated the basic properties of the norm. For instance, he showed that if $Z(G) = 1$, then $K(G) = 1$ (see Corollary 3 of [2]).

More recently in [9], this subgroup has been generalized. The second author defined $B_n(G)$ to be the intersection of all the normalizers of non-n-subnormal subgroups of G with the stipulation that $B_n(G) = G$ if all subgroups of G are n-subnormal. Recall that a subgroup K of G is said to be subnormal in G if there exists a finite series of subgroups of G such that

$$K = K_0 \lhd K_1 \lhd \cdots \lhd K_m = G.$$

* speaker

If K is subnormal in G, then the defect of K in G is the shortest length of such a series. We shall say that a subgroup K of G is n-subnormal if K is subnormal of defect at most n.

Observe that every proper subgroup of G is not 0-subnormal, so $B_0(G)$ is the intersection of the normalizers of all proper subgroups of G, and hence, $B_0(G) = K(G)$, so we may view the $B_i(G)$'s as generalizations of $K(G)$. On the other hand, normal subgroups are precisely the 1-subnormal subgroups of G, so $B_1(G)$ is the intersection of all the non-normal subgroups of G, and as we mentioned above, this implies that $B_1(G) = K(G)$. Clearly, $B_0(G) = B_1(G)$. Zarrin in [9], proved that $B_n(G)$ is a nilpotent characteristic subgroup of G of nilpotence class at most $\mu(n)$, where μ is a function that depends only on n of Roseblade's Theorem see [3]. In particular, $\mu(n)$ is an upper bound on the nilpotence class of a group where all of the subgroups are subnormal of defect at most n. It is known that $\mu(1) = 2$ and $\mu(2) = 3$, but it is an open question as to the values of $\mu(n)$ for $n > 3$.

The present author showed, in the proof of the main theorem of [6], that if a group G has finitely many centralizers, say n (G is a C_n-group), then G is nilpotent-by-finite. This result suggests that the behavior of centralizers has a strong influence on the structure of the group (for more information see [5]-[8]). This is the main motivation to introduce a new series of norms in groups by using normalizers of centralizers.

Definition 1.1. *Let G be a group and $C(G)$ denote the intersection of the normalizers of centralizers of all elements of G. Clearly $B_1(G) \leq C(G)$. Set $C_0(G) = 1$ and define the series whose terms $C_i(G)$ are characteristic subgroups as follows:*

$$C_{i+1}(G)/C_i(G) = C(G/C_i(G))$$

for $i \geq 0$. Denote by $C_\infty(G)$ the terminal term of this ascending series.

We give a characterization of finitely generated nilpotent groups in terms of the subgroups $C_i(G)$, as follows:

Theorem 1.2. *Let G be a finitely generated group. Then the following statements are equivalent:*

1. *G is nilpotent,*
2. *$G = C_n(G)$ for some positive integer n,*
3. *$G/C_m(G)$ is nilpotent for some positive integer m.*

2. Main Results

We denote by $R_n(G)$ the set of all right n-Engel elements of G and for a given positive integer n, a group is called n-Engel if $G = R_n(G)$. For the proof of the main theorem we need the following lemmas.

Lemma 2.1. *For any group G, the subgroup $C(G)$ is nilpotent of class at most 3, and so it is soluble of class at most 2.*

Lemma 2.2. *For every subgroup H of G, we have*

$$H \cap C(G) \leq C(H).$$

We denote by $Z_i(G)$ the i-th term of the upper central series of G. It appears that there exists a close connection between the sequence $C_i(G)$ and the sequences $Z_i(G)$ and $R_i(G)$.

Lemma 2.3. *For any group G, we have*

$$Z_{i+1}(G) \leq C_i(G) \leq R_{2i}(G).$$

The class of nilpotent groups is not closed under forming extensions. However, we have the following well-known result, due to P. Hall (this result is often useful for proving that a group is nilpotent).

Here we prove the following statement (note that the subgroup $C(G)$ is nilpotent).

Proposition 2.4. *For any finitely generated group G, we have*

$$G/C(G) \ \ is \ \ nilpotent \Longleftrightarrow G \ \ is \ \ nilpotent.$$

We obtain the main theorem by the above Lemmas.

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Counting 2-Engelizers in finite groups

Raheleh Khoshtarash* and Mohammad Reza R. Moghaddam

Abstract

In 2016, the second author introduced the notion of 2-Engelizer of the element x in the given group G and denoted the set of all 2-Engelizers in G by $E^2(G)$. In this talk, we classify all non 2-Engel finite groups G, when $|E^2(G)| = 4, 5$.

Keywords and phrases: 2-Engelizer subgroup, 2-Engel element, 2-Engel group.

2010 *Mathematics subject classification:* Primary: 20F45, 20B05; Secondary: 20E07, 20F99.

1. Introduction

In 2016, Moghaddam and Rostamyari [4] introduced the notion of 2-Engelizer of each non-trivial element of G as following

For a given element x of a group G, we call

$$E_G^2(x) = \{y \in G : [x, y, y] = 1, [y, x, x] = 1\}$$

to be the set of all *2-Engelizers* of x in G. The family of all 2-Engelizers in G is denoted by $E^2(G)$ and $|E^2(G)|$ denotes the number of distinct 2-Engelizers in G.

As an example consider $Q_{16} = \langle a, b : a^8 = 1, a^4 = b^4, b^{-1}ab = a^{-1}\rangle$, the Quaternion group of order 16 and take the element b in Q_{16}. Then one can easily check that the 2-Engelizer set of b is as follows:

$$E_{Q_{16}}^2(b) = \{1, a^2, a^4, a^6, b, a^2b, a^4b, a^6b\}.$$

We remark that for the identity element e of G, we have $G = E_G^2(e)$ and hence $G \in E^2(G)$.

* speaker

Clearly in general, the 2-Engelizer of each non-trivial element of a group G does not form a subgroup. (see [4], Example 2.3 for more detail).

Theorem 1.1. *([4], Theorem 2.5) Let G be an arbitrary group. Then the set of each 2-Engelizer of a non-trivial element in G forms a subgroup if and only if the group $x^{E^2_G(x)}$ is abelian for all non-trivial element x of G.*

In the present paper, we talk about the groups G which satisfies the above theorem.

An element x of a group G is called a *right 2-Engel* element, if for every $y \in G$, $[x, {}_2\, y] = [x, y, y] = 1$, and the set of all right 2-Engel elements of G is denoted by $R_2(G)$.

The following lemmas play important role in finding lower bound for $|E^2(G)|$.

Lemma 1.2. *Let G be a group. Then $R_2(G)$ is the intersection of all 2-Engelizers in G.*

Lemma 1.3. *A group G is the union of 2-Engelizers of all elements in $G \backslash R_2(G)$, that is to say $G = \bigcup_{x \in G \backslash R_2(G)} E^2_G(x)$.*

Lemma 1.4. *Let $|E^2_{G/R_2(G)}(xR_2(G))| = p$, for some non right 2-Engel element x of a group G and a prime number p. For all $y \in G \backslash R_2(G)$, if $E^2_{G/R_2(G)}(xR_2(G)) = E^2_{G/R_2(G)}(yR_2(G))$ then*

$$E^2_G(x) = E^2_G(y).$$

2. Main Results

In this section, we compute $|E^2(G)|$ under certain conditions on G.

Theorem 2.1. *Let G be a group such that $G/R_2(G) \cong C_p \times C_p$, for any prime number p. Then $|E^2(G)| = p + 2$.*

In 1970, Bruckhheimer, Bryan and Muir [3] showed the following result, which is useful in our further investigations.

Theorem 2.2. *([3], Theorem 1) A group G is the non-trivial union of three subgroups if and only if it is homomorphic to the Klein four group.*

Now, using the above theorem we have the following result.

Theorem 2.3. *Let G be a group, then $|E^2(G)| = 4$ if and only if $G/R_2(G) \cong C_2 \times C_2$.*

To prove our final result we need the following lemma.

Lemma 2.4. *Let E^2_1, E^2_2, E^2_3, and E^2_4 be the four proper 2-Eengelizers of a group G. Then*
(a) none of them is contained in the union of the others;
(b) no element of G is in exactly two or three of E^2_i's, $1 \leq i \leq 4$.

Theorem 2.5. *Let G be a finite group. If $|E^2(G)| = 5$, then $G/R_2(G) \cong C_3 \times C_3, D_{12}, C_3 \rtimes C_4, C_2 \times C_6, A_4$ or S_3.*

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h-conditionally permutable subgroups and PST-groups

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Abstract

A subgroup H of a finite group G is said to be h-conditionally permutable in G if for every Hall subgroup X of G, there exists an element $g \in G$ such that $HX^g = X^gH$. In this paper, we obtain some results on h-conditionally permutable and by using this results, we are able to determine some new characterizations of solvable PST-groups.

Keywords and phrases: h-conditionally permutable subgroups, h-conditionally semipermutable subgroups, PST-groups..

2010 *Mathematics subject classification:* Primary: 22D10; Secondary: 20D20.

1. Introduction

An iteresting question in finite group theory is to determine the influence of the embedding properties of members of some distinguished families of subgroups on the structure of the group. The present paper adds some result to this line of research.

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover p is always supposed to be a prime and π is non-empty subset of all primes. We use $\pi(G)$ to denotes a set of primes divisor of $|G|$ and $G^{\mathfrak{N}}$ the nilpotent residual of G. Given a group G, two subgroups H and K of G are said to permute if $HK = KH$, that is, HK is a subgroup of G .A subgroup H of G is called permutable in G if H permutes with all subgroups of G. Some weaker concepts than that of permutability were also studied. A subgroup H of G is called S-permutable in G if H permutes with all Sylow subgroups of G. This concept was first introduced by Kegel in [5] who called them S-quasinormal.

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As a generalizations of permutability and S-permutability, Zhongmu Chen in [2] introduced the following: A subgroup H of G is called semipermutable in G if H permutes with all subgroups K of G with $(|H|, |K|) = 1$, and S-semipermutable if H permutes with all Sylow p-subgroups of G with $(p, |H|) = 1$, who called them seminormal, S-seminormal subgroups of G. respectively. A group G is called a PT-group if permutability is a transitive relation and PST-group if S-permutability is a transitive relation. By Kegel's result [5], a group G is a PST-group if and only if every subnormal subgroup of G is S-permutable in G. In 1975, Agrawal [1] showed that a solvable group G is a PST-group if and only if the nilpotent residual L of G is a normal abelian Hall subgroup of G upon which G acts by conjugation as power automorphisms.
For two subgroups H and K of G, we often meet the situation $HK \neq KH$, but there exists an element $x \in G$ such that $HK^x = K^xH$. Guo et all. [3], introduced the concept of conditionally permutable subgroups as follows: A subgroup H of a group G is called conditionally permutable in G if for every subgroup K of G, there exists some $x \in G$ such that $HK^x = K^xH$. As a generalization of conditionally permutable subgroups, Hung and Guo [4] introduced the following definition.

Definition 1.1. *A subgroup H of a group G is called s-conditionally permutable in G if for every Sylow subgroup X of G, there exists an element $g \in G$ such that $HX^g = X^gH$.*

By Sylow's theorem we see that a subgroup H of a group G is s-conditionally permutable in G if and only if for every $p \in \pi(G)$, there exists at least one Sylow p-subgroup X of G such that $HX = XH$.

By using s-conditionally permutable, S-semipermutable and Hall subgroups, we introduce some new concepts as follows.

Definition 1.2. *A subgroup H of G is called h-conditionally permutable in G if for every Hall subgroup X of G with $(|H|, |X|) = 1$, there exists an element $g \in G$ such that $HX^g = X^gH$.*

It is clear that a subgroup H of a solvable group G is h-conditionally permutable in G if and only if for every $\pi \subseteq \pi(G)$, there exists at least one Hall π-subgroup X of G such that $HX = XH$.

Definition 1.3. *A subgroup H of G is called h-conditionally semipermutable in G if for every Hall subgroup X of G with $(|H|, |X|) = 1$, there exists an element $g \in G$ such that $HX^g = X^gH$.*

Using the concepts h-conditionally permutable and h-conditionally semipermutable subgroups, some new characterizations of finite groups are obtained and several interesting results are generalized in this paper. First of all, we give some basic results which are needed in the sequel.

Lemma 1.4. *Suppose that a subgroup H of a solvable group G is h-conditionally permutable in G and $N \trianglelefteq G$. Then HN/N is h-conditionally permutable in G/N.*

Lemma 1.5. *Suppose that a p-subgroup H of a solvable group G is h-conditionally semipermutable in G and let N be a normal q-subgroup of G, where $p, q \in \pi(G)$. Then HN/N is h-conditionally semipermutable in G/N.*

Lemma 1.6. *Suppose that H is a p-subgroup of a group G, where $p \in \pi(G)$. Then H is h-conditionally permutable in G if and only if H is h-conditionally semipermutable in G.*

2. Main Results

In this paper, we study the properties of h-conditionally permutable and h-conditionally semipermutable subgroups and we will generalize some earlier results.

Theorem 2.1. *Let G be a solvable group and for any $p \in \pi(G)$, every p-subgroup of G, h-conditionally permutable in G. Then G is supersolvable.*

Corollary 2.2. *Let G be a solvable group and for any $p \in \pi(G)$, every p-subgroup of G, h-conditionally semipermutable in G. Then G is supersolvable.*

Theorem 2.3. *Let G be a solvable group and every subnormal subgroup of G, s-conditionally permutable in G. Then G is supersolvable.*

Theorem 2.4. *Let G be a solvable group and q the largest prime dividing $|G|$. Then for any $p \in \pi(G)$, every p-subgroup of G is h-conditionally permutable in G if and only if G has a normal Hall subgroup N such that for any $p \in \pi(G)$, every p-subgroup of G/N is h-conditionally permutable in G/N, furthermore, if $q \nmid |N|$, for any $p \in \pi(G)$, every p-subgroup of N is normal in G, if $q \mid |N|$, for any $p \in \pi(G)$ with $p \neq q$, every p-subgroup of N is normal in G, and if $q \mid |N|$, for any q-subgroup H of N and for any Hall q'-subgroup L of G, there exists a subgroup K of G such that $K \leq N_G(H)$ and $L^g \leq K$ for some $g \in G$.*

Corollary 2.5. *Let G be a solvable group and q the largest prime dividing $|G|$. Then for any $p \in \pi(G)$, every p-subgroup of G is h-conditionally semipermutable in G if and only if G has a normal Hall subgroup N such that for any $p \in \pi(G)$, every p-subgroup of G/N is h-conditionally semipermutable in G/N, furthermore, if $q \nmid |N|$, for any $p \in \pi(G)$, every p-subgroup of N is normal in G, if $q \mid |N|$, for any $p \in \pi(G)$ with $p \neq q$, every p-subgroup of N is normal in G, and if $q \mid |N|$, for any q-subgroup H of N and for any Hall q'-subgroup L of G, there exists a subgroup K of G such that $K \leq N_G(H)$ and $L^g \leq K$ for some $g \in G$.*

Corollary 2.6. *Let G be a solvable group. Then every Sylow subgroup of G is h-conditionally permutable in G if and only if G has a normal Hall subgroup N such that every Sylow subgroup of G/N is h-conditionally permutable in G/N and every Sylow subgroup of N is normal or h-conditionally permutable in G.*

Corollary 2.7. *Let G be a solvable group. Then every Sylow subgroup of G is h-conditionally semipermutable in G if and only if G has a normal Hall subgroup N such that every Sylow subgroup of G/N is h-conditionally semipermutable in G/N and every Sylow subgroup of N is normal or h-conditionally semipermutable in G.*

From 1950s, especially in recent years, due to the efforts of many leading mathematicians, such as Gaschüatz, Robinson, Cossey, Ballester-Bolinches, ect, many characterizations of PST-group were discovered. The study of these classes of groups has undoubtedly constituted a fruitful topic in group theory. Now we prove the following theorem which introduce some new characterizations of solvable PST-groups.

Theorem 2.8. *Let G be a solvable group and q the largest prime dividing $|G|$. Suppose further that Q is the sylow q-subgroup of G and $G^{\mathfrak{N}} \neq Q$. Then the following conditions are equivalent:*
(i) Every subnoamal subgroup of G is h-conditionally permutable in G.
(ii) Every subnoamal subgroup of G is s-conditionally permutable in G.
(iii) G is a PST-group.

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Split Prime and Solvable Graphs

J. Mirzajani* and A. R. Moghaddamfar

Abstract

The prime (resp. solvable) graph GK(G) (resp. $\mathcal{S}(G)$) of a finite group G is a simple graph whose vertices are the prime divisors of the order of G and two distinct vertices p and q are joined by an edge if and only if G has a cyclic (resp. solvable) subgroup of order divisible by pq. In this talk, we first show that the prime graph of any alternating and sporadic simple groups is split, that is, a graph whose vertex set can be partitioned into two sets such that the induced subgraph on one of them is a complete graph and the induced subgraph on the other is an independent set. Next, we prove that the solvable graph of any alternating and sporadic simple groups is split, except for the following simple groups: M_{22}, M_{23}, M_{24}, Co_3, Co_2, Fi_{23}, Fi'_{24}, B, M and J_4. Finally, we consider the compact form of a prime graph and show that the compact form of a nonabelian simple group is split.

Keywords and phrases: Prime graph, solvable graph, split graph, finite simple group.

2010 *Mathematics subject classification:* 20D05, 20D06, 20D08, 20D10.

1. Notation and Definitions

The graphs will be considered here are finite, simple and undirected. Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a graph with vertex set V_Γ and edge set E_Γ. Given a set of vertices $X \subseteq V_\Gamma$, the subgraph induced by X is written $\Gamma[X]$. A subset of vertices often is identified with its induced subgraph, and vice versa. A subset $X \subseteq V_\Gamma$ is called a *clique* if the induced subgraph $\Gamma[X]$ is complete, and it is *independent* if $\Gamma[X]$ is a null graph. A graph Γ is called a *split* graph if its vertex set has a partition $V_\Gamma = C \uplus I$, where C is a complete and I an independent set, any such partition will be called a *split partition*. Moreover, a split partition $V_\Gamma = C \uplus I$ is *special* if every vertex in I is

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not adjacent to at least one vertex in C. Note that, a split partition (or a special split partition) of a split graph is not unique, but it is always possible to choose a partition such that C is a clique of maximum size. Split graphs were introduced already in 1977 by Földes and Hammer [3].

We will consider only finite groups herein. The set of all prime divisor of $|G|$ is denoted by $\pi(G)$ which is called *prime spectrum* of G. In the *prime graph* of G, which is denoted by $\mathrm{GK}(G)$, the vertex set is the prime spectrum of G and two distinct vertices p and q are joined by an edge (and we shortly write $p \sim q$) if and only if G has an element of order pq. We denote by $s(G)$ the number of connected component of $\mathrm{GK}(G)$, and by $\pi_i(G)$, $i = 1, 2, \ldots, s(G)$, the vertex set of ith connected component of $\mathrm{GK}(G)$. We recall that the vertex set of prime graphs of all finite simple groups are determined in [4] and [7]. Similarly, the *solvable graph* of a group G, denoted by $\mathcal{S}(G)$, is a simple graph with vertex set $\pi(G)$ and two vertices p and q are joined by an edge (and we shortly write $p \approx q$) if and only if G has a solvable subgroup whose order is divisible by pq. Solvable graphs were introduced as a generalization of prime graphs in [1]. As a matter of fact, if $p \sim q$ in $\mathrm{GK}(G)$, then $p \approx q$ in $\mathcal{S}(G)$. This shows that $\mathrm{GK}(G)$ is a subgraph of $\mathcal{S}(G)$. For a natural number n, we denote by l_n the largest prime not exceeding n and by s_n denote the smallest prime greater than n.

2. Results

The ith connected component of $\mathrm{GK}(G)$ is denoted by $\mathrm{GK}_i(G) = (\pi_i(G), E_i(G))$, $i = 1, 2, \ldots, s(G)$.

Theorem 2.1. [5, Suzuki] *Let G be a finite simple group whose prime graph $\mathrm{GK}(G)$ is disconnected and let $\mathrm{GK}_i(G)$ be a connected component of $\mathrm{GK}(G)$ with $i \geqslant 2$. Then $\mathrm{GK}_i(G)$ is a clique.*

The following result is taken from [3].

Theorem 2.2. (Forbidden Subgraph Characterization) *A graph is a split graph if and only if it contains no induced subgraph isomorphic to $2K_2$ (two parallel edges), C_4 (a square), or C_5 (a pentagon).*

Corollary 2.3. *Let Γ be a split graph with a split partition $V = C \uplus I$. If Γ has more than one connected component, then the connected components consist of the following possibilities: a complete subset of V containing C (which is a clique) and $\{v$, a single prime$\} \subseteq I$.*

The adjacency criterion for two vertices in the prime graph of an alternating group is obvious and can be stated as follows (see [6]).

Theorem 2.4. *Let $n \geqslant 5$ be a natural number and let p and q be two distinct odd primes in $\pi(\mathbb{A}_n)$. Then, we have:*

(1) $p \sim q$ *in* $\mathrm{GK}(\mathbb{A}_n)$ *if and only if* $p + q \leqslant n$.

(2) $2 \sim p$ *in* $\mathrm{GK}(\mathbb{A}_n)$ *if and only if* $4 + p \leqslant n$.

The adjacency criterion for two vertices in the prime graph of a symmetric group is similar: two distinct primes p and q in $\pi(\mathbb{S}_n)$ are adjacent in $\mathrm{GK}(\mathbb{S}_n)$ if and only if $p + q \leqslant n$.

The following result is immediate from Theorem 2.4.

Theorem 2.5. *The prime (solvable) graph of any alternating group $\mathbb{A}_n$, $n \geqslant 5$, is a split graph with a split partition $C \uplus I$, where $C = \{2, 3, 5, \ldots, l_{\lfloor \frac{n}{2} \rfloor}\}$ and $I = \{s_{\lfloor \frac{n}{2} \rfloor}, \ldots, l_n\}$.*

Similarly, we have the following theorem.

Theorem 2.6. *The prime (solvable) graph of every symmetric group $\mathbb{S}_n$ is a split graph with a split partition $C \uplus I$, where $C = \{2, 3, 5, \ldots, l_{\lfloor \frac{n}{2} \rfloor}\}$ and $I = \{s_{\lfloor \frac{n}{2} \rfloor}, \ldots, l_n\}$.*

The information which are needed for determining the structure of prime graphs and solvable graphs of sporadic simple groups can be found in [2].

Theorem 2.7. *The solvable graph of any sporadic simple group is a split graph, except following cases: M_{22}, M_{23}, M_{24}, Co_3, Co_2, Fi_{23}, Fi'_{24}, B, $M = F_1$ and J_4.*

Given a graph Γ, we define the equivalence relation on the vertex set V_Γ of Γ putting $u \equiv v$ if and only if u and v are adjacent and have the same neighbourhood for every $u, v \in V_\Gamma$. Denote by $\Gamma_c = \Gamma/_{\equiv}$ the quotient graph with respect to this equivalence and call it the *compact form* of Γ. We will denote by $\mathrm{GK}_c(G)$ and $\mathcal{S}_c(G)$ the compact form of prime graph and solvable graph of a group G.

Corollary 2.8. *The compact forms $\mathrm{GK}_c(G)$ and $\mathcal{S}_c(G)$ of any alternating and sporadic simple group are split.*

Theorem 2.9. *If G is a simple group of Lie type, then $\mathrm{GK}_c(G)$ is split.*

Corollary 2.10. *The compact form $\mathrm{GK}_c(G)$ of a nonabelian simple group G is split.*

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Some symmetric designs invariant under the small Ree groups

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Abstract

A construction of primitive 1-designs invariant under the action of a finite primitive group is used to determine the parameters of certain symmetric 1-designs. Using a method referred to as Key-Moori Method 1, in this paper we construct a number of symmetric 1-designs from the maximal subgroups of the small Ree groups ${}^2G_2(q)$.

Keywords and phrases: small Ree groups, designs, conjugacy class, maximal subgroup, primitive permutation representation.

2010 *Mathematics subject classification:* Primary: 20D05, 05E15 ; Secondary: 05E20.

1. Introduction

In [2] and [3], two methods for constructing combinatorial designs and codes from finite simple groups, were introduced by Key and Moori. In the first method [2], hereafter called Key-Moori method 1, the authors considered primitive permutation representations of a finite simple group to construct symmetric 1-designs. The aim of the present paper is to construct designs using Key-Moori Method 1, from the maximal subgroups of the family of small Ree groups ${}^2G_2(q)$, where q is an odd power of 3. Our notation for designs is as in [1]. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an incidence structure, i.e. a triple with point set $\mathcal{P}$, block set $\mathcal{B}$ disjoint to $\mathcal{P}$ and incidence set $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$. If the ordered pair $(p, B) \in \mathcal{I}$ we say that p is incident with B. It is often convenient to assume that the blocks in $\mathcal{B}$ are subsets of $\mathcal{P}$ so $(p, B) \in \mathcal{D}$ if and only if $p \in B$. For a positive integer t, we say that $\mathcal{D}$ a t-design if every block $B \in \mathcal{B}$ is incident with exactly k points and

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every t distinct points are together incident with λ blocks. In this case we write $\mathcal{D} = t\text{-}(v, k, \lambda)$ where $v = |\mathcal{P}|$. We say that $\mathcal{D}$ is symmetric if it has the same number of points and blocks.

In this paper, using Key-Moori Method 1, which we restate in Section 3 (Lemma 3.1), we construct a number of symmetric 1-designs from the maximal subgroups of the small Ree groups ${}^2G_2(q)$. We achieve this in Section 3 by proving a series of lemmas and propositions that lead to Theorem 3.6. In Section 2 we recall some known results and produce new results of Ree groups.

2. Some information on Ree groups

The aim of this section is to assemble in readily usable form a collection of facts and results concerning the small Ree groups and which will be used in the sequel. The small Ree groups, denoted ${}^2G_2(q)$ where q is an odd power of 3, are a family of groups discovered by Rimhak Ree in the 60's [5]. He showed that these groups are simple except for the first one ${}^2G(3)$, which is isomorphic to $PSL_2(8){:}3$. In [7], Wilson presented a simplified construction of the Ree groups, as an automorphism of a 7-dimensional vector space over the field of q elements. Let $G =^2 G_2(q)$ be a small Ree group (we always assume that $3^{2n+1} = q \geq 27$ to avoid the non-simple case). The order of G is $q^3(q^3+1)(q-1)$ and G acts doubly transitive on a set Ω of size q^3+1, more precisely acts as automorphisms of a 2-$(q^3+1, q+1, 1)$ block design. For a group G, we denote the set of all involutions of G by $\mathcal{J}(G)$ and the set of all Sylow r-subgroups of G by $\mathcal{P}_r(G)$.

Lemma 2.1. *Let U_p be a Sylow p-subgroup of G. Then*

(i) *for $p > 3$, U_p is cyclic;*

(ii) *each pair of Sylow 3-subgroups of G have trivial intersection;*

(iii) $N_G(U_2) \cong U_2{:}7{:}3$ *and* $C_G(U_2) = U_2 \cong 2^3$*;*

(iv) *for* $\{1_G\} \neq S \leq U_3$ *we have* $N_G(S) \leq N_G(U_3)$*;*

(v) *if* $Q_1 \leq G$ *and* $Q_1 \cong 2^2$*, then* $C_G(Q_1) \cong Q_1 \times D_{\frac{q+1}{2}}$*;*

(vi) *each pair of 2-subgroups of G of equal order are conjugate in G.*

Theorem 2.2. *Up to conjugacy, the maximal subgroups of G are of shapes:*

(1) $q^{1+1+1}{:}C_{q-1}$*;*

(2) $C_{q-\sqrt{3q}+1}{:}C_6$*;*

(3) $C_{q+\sqrt{3q}+1}{:}C_6$*;*

(4) $2 \times PSL_2(q)$*;*

(5) $(2^2 \times D_{\frac{q+1}{2}}){:}3$*;*

(6) ${}^2G_2(q_0)$*, where* $q_0 = \sqrt[r]{q}$*, for a prime r.*

Notice that the maximal subgroups of shape $C_{q\pm\sqrt{3q}+1}{:}C_6$ are Frobenius groups.

Notation. We use the following notation for the rest of the paper. Let $t_1 = (q-1)/2$, $t_2 = (q+1)/4$, $t_3 = q - \sqrt{3q} + 1$, $t_4 = q + \sqrt{3q} + 1$ and $t_5 = q^3$. Also set $\{t_1, t_2, t_3, t_4, t_5\}$. For $1 \le i \le 5$, we denote by $\mathcal{B}_i$ the set of all subgroups of G of order t_i.

Proposition 2.3. *Let $\mathcal{B}_i$ be as above, and suppose that $B_i \in \mathcal{B}_i$ are chosen arbitrarily for $1 \le i \le 5$. Then*

(i) *every element of $\mathcal{B}_i$ is a Hall subgroup of G; in particular every two elements of $\mathcal{B}_j$ for a fixed j are conjugate in G;*
(ii) $N_G(B_1) = B_1{:}2 \cong D_{2(q-1)}$;
(iii) $N_G(B_2) = (Q_1 \times (B_2{:}2)){:}3$ *and* $N_G(Q_1) = N_G(B_2)$, *where* $Q_1 \cong 2^2$;
(iv) $N_G(B_3) = B_3{:}6$;
(v) $N_G(B_4) = B_4{:}6$;
(vi) $N_G(B_5) = B_5{:}(q-1)$;
(vii) *if $i \ne 5$ then B_i is cyclic; and if $\{1_G\} \ne S \le B_j$ then $N_G(S) = N_G(B_j)$.*

PROOF. All parts follow from [6]. □

A Sylow 3-subgroup P of G is a TI-subgroup, i.e. for $g \notin N_G(P)$, we have $P \cap P^g = \{1_G\}$. The group P is a 3-group of order q^3 of nilpotence class 3 with $|Z(P)| = q$ and $|P'| = q^2$. Both P' and P/P' are elementary abelian 3-groups. Moreover, all elements of order 3 lie in P'. All non-identity elements of $Z(P)$ are conjugate in G and

$$\bigcup_{P \in \mathcal{P}_3(G)} (P' \setminus Z(P)) = b^G \cup (b^{-1})^G, \tag{1}$$

for $b \in P' \setminus Z(P)$. Also we have $P \setminus P'$ is a union of three conjugacy classes of G.

Lemma 2.4. *Let M be a maximal subgroup of G of shape $2 \times PSL_2(q)$. If P_1 is a Sylow 3-subgroup of M then $N_G(P_1) \cong P'{:}(q-1)$ where P is the Sylow 3-subgroup of G containing P_1.*

3. Constructing designs using Method 1

In this section, we construct symmetric 1-designs from some primitive permutation representations of the small Ree groups. For a finite group G and a maximal subgroup M of G, we define $\mathcal{A}_M = \{|M \cap M^g| | g \in G\}$. It is clear that $\mathcal{A}_M \ne \emptyset$, for $|M| \in \mathcal{A}_M$. We use this notation for the remainder of the paper. Our method for constructing these designs is based on the following result.

Lemma 3.1. ***(Method 1)*** *Let G be a finite primitive permutation group acting on the set Ω of size n. Let $\alpha \in \Omega$, and let $\Delta \neq \{\alpha\}$ be an orbit of the stabilizer G_α of α. If*

$$\mathcal{B} = \{\Delta^g : \ g \in G\},$$

then $\mathcal{B}$ forms a 1-$(n, |\Delta|, |\Delta|)$ design with n blocks, with G acting as an automorphism group on this structure, primitive on the points and blocks of the design.

Proof. See [2, Proposition 1]. □

Lemma 3.2. *Let G be a finite simple group acting on the set of conjugates of a maximal subgroup M by conjugation. Then the sizes of orbits of a point stabilizer of G are given as elements of the set*

$$\left\{ \frac{|M|}{m} : m \in \mathcal{A}_M \right\}.$$

Proof. See [4, Lemma 3.3]. □

Remark 3.3. *Let $\mathcal{D}(m)$ be a triple consisting of the parameters of designs constructed from a primitive permutation representation of M and a suborbit of size $k = \frac{|M|}{m}$ (recall that a suborbit is an orbit of the action of a point stabilizer). Then $\mathcal{D}(m)$ are 1-$(|G : M|, \frac{|M|}{m}, \frac{|M|}{m})$ designs.*

If M is a maximal subgroup of G of shape $q^{1+1+1}{:}C_{q-1}$, then the following lemma shows that the designs constructed from M using Key-Moori Method 1 are trivial.

Lemma 3.4. *Let M be a maximal subgroup of G of shape $q^{1+1+1}{:}C_{q-1}$. Then $\mathcal{A}_M = \{q-1, |M|\}$.*

Proposition 3.5. *Let $M \cong 2{\times}PSL_2(q)$ be a maximal subgroup of G. Then $\mathcal{A}_M = \{1, 2, 4, q, 2(q+1), |M|\}$.*

Proof. First we show that $\mathcal{A}_M \subseteq \{1, 2, 4, q, 2(q+1), |M|\}$. It is clear that $|M| \in \mathcal{A}_M$. So assume that $g \in G \setminus M$ and $M = BC$ where $B \cong PSL_2(q)$, $|C| = 2$ and $[B, C] = \{1_G\}$. Also suppose that $M = C_G(j)$ for an involution j. We put $L := M \cap M^g$. Our goal is to prove that $|L| \in \{1, 2, 4, q, 2(q+1)\}$. We have divided the proof into several steps.

Step 1: We have $gcd(|L|, \frac{q-1}{2}) = 1$.

Step 2: If $j \in L$ then $|L| = 2(q+1)$.

Step 3: If $gcd(|L|, \frac{q+1}{4}) \neq 1$ then $|L| = 2(q+1)$.

Step 4: If $3 \| |L|$, then $q \| |L|$.

Step 5: $|L| \neq 8$.

Step 6: $|L| \neq 2q$.

Step 7: $|L| \in \{1, 2, 4, q, 2(q+1)\}$.

Observe that up to now we have proved that $\mathcal{A}_M \subseteq \{1, 2, 4, q, 2(q+1), |M|\}$. Hence, we need to prove that the converse also holds. For this, we divide the proof into the following steps:

Step $1'$**:** $2 \in \mathcal{A}_M$.

Step $2'$**:** $4 \in \mathcal{A}_M$.

Step $3'$**:** $2(q+1)$ lies in $\mathcal{A}_M$.

Step $4'$**:** q lie in $\mathcal{A}_M$.

Step $5'$**:** $1 \in \mathcal{A}_M$.

Assume that

$$T_2 := \{g \in G \mid |M \cap M^g| \textit{ is even} \} \text{ and } T_3 := \{g \in G \mid M \cap M^g \in \mathcal{P}_3(M)\}.$$

Our aim is to prove that $T := T_2 \cup T_3$ is strictly contained in G. Suppose that $g \in T_3$. As all Sylow 3-subgroups in M are conjugate, it is easy to see that $T_3 \subseteq \bigcup_{P \in \mathcal{P}_3(M)} MN_G(P)$. Therefore $|T_3| \leq \sum_{P \in \mathcal{P}_3(M)} |MN_G(P)|$. Fix an element $P \in \mathcal{P}_3(M)$. By Lemma 2.4 and its proof, we have $|N_G(P)| = q^2(q-1)$ and $|N_M(P)| = q(q-1)$. Since G contains $q+1$ Sylow 3-subgroups, we have

$$|T_3| \leq (q+1)|M : N_M(P)||N_G(P)| = q^2(q-1)(q+1)^2.$$

Now we find a bound for T_2. Assume that $j', j'' \in \mathcal{J}(M)$. So we have

$$|C_G(j', j'')| = |C_G(j')| = q(q-1)(q+1).$$

Also it is to check that $|\mathcal{J}(M)| = q(q-1)+1$. Hence we can write

$$|T_2| \leq |\{g \in G \mid j'^g = j'' \text{ for some } j', j'' \in \mathcal{J}(M)\}| \leq (q(q-1)+1)^2 q(q^2-1).$$

Therefore,

$$|T| = |T_2 \cup T_3| \leq |T_2| + |T_3| \leq (q(q-1)+1)^2 q(q^2-1) + q^2(q^2-1)(q+1) < |G|.$$

If we choose an element $g \in G \setminus T$, then it is easy to see that $M \cap M^g = \{1_G\}$ and the result follows.

□

The following result is an immediate consequent of Lemma 3.2 and Proposition 3.5.

Theorem 3.6. *Let M be a maximal subgroup of G of type $2 \times PSL_2(q)$. Then the parameters of designs constructed M using Method 1 are as follows.*

- $\mathcal{D}(1) = \big(q^2(q^2 - q + 1), q(q-1)(q+1), q(q-1)(q+1)\big)$.
- $\mathcal{D}(2) = \big(q^2(q^2 - q + 1), q(q-1)(q+1)/2, q(q-1)(q+1)/2\big)$.
- $\mathcal{D}(4) = \big(q^2(q^2 - q + 1), q(q-1)(q+1)/4, q(q-1)(q+1)/4\big)$.
- $\mathcal{D}(q) = \big(q^2(q^2 - q + 1), (q-1)(q+1), (q-1)(q+1)\big)$.
- $\mathcal{D}(2(q+1)) = \big(q^2(q^2 - q + 1), q(q-1)/2, q(q-1)/2\big)$.

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The Structure of finite groups with trait of non-normal subgroups

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Abstract

In this paper, we classify all the finite groups all of whose non-normal nilpotent subgroups are cyclic. We show that such groups are solvable with cyclic centers. If G is a non-supersolvable group, then G has only one non-cyclic Sylow subgroup which is either isomorphic to Q_8 or is of type (q, q).

Keywords and phrases: Non-normal subgroups; Conjugace Class; Non-nilpotent groups..

2010 *Mathematics subject classification:* Primary: 20D99; Secondary: 20E45.

1. Introduction

Finding the structure of the groups by applying the features of their subgroups is the subject of many researches in group theory. One of the most natural of these features is the cyclicness of the subgroups. In [1, theorem 2.16] the authors classify the finite p-groups all of whose non-normal subgroups are cyclic and in [3] the authors classify finite p-groups all of whose non-normal abelian subgroups are cyclic. However, no investigation has been made in the non-nilpotency case.

If all the Sylow subgroups of G are cyclic, then by [2, theorem 5.16] we have $G = G'Y$ where G' is a cyclic Hall subgroup and Y is cyclic too. In this case if H is a non-normal nilpotent subgroup of G, then we can write $H = LK$ where $K \leqslant G'$, $L \leqslant Y$ and $(|L|, |K|) = 1$. The nilpotency of H implies that $[K, L] = 1$ and so H is cyclic. Therefore, every non-normal nilpotent subgroup of G will be cyclic. But the converse does not hold, that is, if all the non-normal nilpotent subgroups of G are cyclic, then its Sylow subgroups are not necessarily cyclic.

* speaker

Assume that the group G has the property that all of its non-normal nilpotent subgroups are cyclic. If G is nilpotent, then all the non-normal subgroups will be cyclic. Let H be a non-normal p-subgroup, for the prime number p. Then $HO_{p'}(G)$ is non-normal subgroup of G and so is cyclic; hence $O_{p'}(G)$ is a cyclic. Therefore, $G \cong P \times O_{p'}(G)$, where all the non-normal subgroups of P are cyclic. The structure of such p-groups is presented in [1, theorem 2.16].

In this paper, we investigate the structure of finite non-nilpotent groups whose non-normal nilpotent subgroups are cyclic and have at least one non-cyclic Sylow subgroup. Our notation is standard and can be found in e.g.[2].

2. Main Results

Lemma 2.1. *Let G be a finite solvable group and P be a non-normal Sylow subgroup of G. If $N_G(P)'$ is a Hall subgroup and $N_G(P)$ is contained in a maximal subgroup of prime index, then G is p-nilpotent.*

Theorem 2.2. *Let G be a non-nilpotent group such that all the non-normal nilpotent subgroups of G are cyclic. Then G is solvable with a cyclic center.*

Lemma 2.3. *Let G satisfy the hypothesis $\mathcal{H}$ and P be a non-normal Sylow subgroup of G. Then all the Sylow subgroups of $N_G(P)$ are cyclic, so $N_G(P)$ is meta-cyclic and $N_G(P)'$ is Hall subgroup of G.*

Lemma 2.4. *Let $G = QM$ satisfy the hypothesis $\mathcal{H}$ and M be a maximal non-normal subgroup of G. If $Q \ncong Q_{2^n}$ and $Q \cap M \neq 1$, then M is of prime index.*

Theorem 2.5. *Let G satisfy the hypothesis $\mathcal{H}$ and M be a maximal subgroup of G that is not of prime index. Also Suppose that Q is a q-subgroup for some prime q such that $Q \nleq M$. Then $G = QM$ and Q is a normal Sylow subgroup of G. Furthermore*

(1) if $Q \cap M = 1$, then $Q \cong \mathbb{Z}_q \times \mathbb{Z}_q$ and M is meta-cyclic with cyclic Sylow subgroups. Also $G' = QM'$ and $C_M(Q)$ is normal and cyclic. If M is nilpotent, then M is cyclic and $Z(G) = C_M(Q)$;

(2) if $Q \cap M \neq 1$, then $|G|$ is even, $Q \cong Q_8$ is the Sylow 2-subgroup of G. Also just one of the Sylow subgroups of G is non-normal which is of order power of 3. Therefore $G \cong (K \times Q_8) \rtimes \mathbb{Z}_{3^m}$, where K is the normal cyclic $\{2,3\}'$-Hall subgroup of G. If M is nilpotent, then $G \cong K \times Q_8 \rtimes \mathbb{Z}_{3^m}$, where $G/Z(G) \cong A_4$.

Remark 2.6. *According to the previous Theorem, if G satisfy the hypothesis $\mathcal{H}$ and has a maximal subgroup that is not of prime index, then we can characterized structure of G. So we can assume that every maximal subgroup of G is of prime index, hence we can assume that G is a supersolvable group (by the well-known theorem of Huppert).*

Theorem 2.7. *Let the supersolvable group G satisfy the hypothesis $\mathcal{H}$. If $M \ntrianglelefteq G$ is a nilpotent maximal subgroup, then $G = G'M$, M is a cyclic and G' is of prime order.*

Theorem 2.8. *Let the supersolvable group G satisfy the hypothesis $\mathcal{H}$ and all the non-normal maximal subgroups of G be non-nilpotent. If P is a non-normal Sylow subgroup of G, such that $M = N_G(P)$ is a maximal subgroup of G, then $G = TM$, where T is a subgroup of order $p \neq q$, $q \geqslant 7$, $G' \leqslant QM'$ where $Q \in \mathcal{S}y\ell_q(G)$, and the inequality is proper when $Q \cap M \leqslant Z(M)$, in which case $G' = TQ'M'$. Furthermore,*

(1) $Q = T(Q \cap M)$ and Q is either isomorphic to $M(p^{n+1})$ or is abelian of type (q^n, q);

(2) any subgroup of Q containing $\Omega_1(Q)$ is normal in G;

(3) all the Sylow subgroups of M are cyclic, $P \leqslant Z(M)$, $M' \trianglelefteq G$ is a cyclic Hall subgroup, and also M/M' is cyclic too;

(4) $L \trianglelefteq G$ if and only if $[\Omega_1(Q), L] = 1$, for any nilpotent q'-subgroup L of G.

Theorem 2.9. *Let the supersolvable group G satisfy the hypothesis $\mathcal{H}$ and the normalizer of any Sylow subgroup of G is not maximal. If P is a non-normal Sylow subgroup of G such that $P \lneqq N_G(P)$, then every non-normal Sylow subgroup of G is contained in $C_G(P)$. Furthermore,*

(1) $G = NH$ where $N \trianglelefteq G$ is a nilpotent Hall subgroup, H is cyclic and all the Sylow subgroups of H are normal in G;

(2) $C_N(K)$ is cyclic for any non-normal subgroup K of H;

(3) The Sylow 2-subgroup of G does not have Quaternion structure. Therefore, if N is Hamiltonian, then it is abelian;

(4) if $L \leqslant N$ is non-normal in G, then for some prime r, $O_r(L) \ntrianglelefteq G$ is a cyclic subgroup and $O_{r'}(N)$ is cyclic too. Also, the Sylow r-subgroup of G is the only non-cyclic Sylow subgroup of G.

Theorem 2.10. *Let the supersolvable group G satisfy the hypothesis $\mathcal{H}$ and the normalizer of any Sylow subgroup of G be non-maximal. If for any non-normal Sylow subgroup P of G,*

$P = N_G(P)$, then $G = PG'$ and $G' = O_{p'}(G)$ is nilpotent. Also for any non-cyclic subgroup L of G' and any subgroup K of P, $C_K(L) \trianglelefteq G$. Furthermore, if $L \leqslant G'$ is non-normal in G, then $O_q(L) \ntrianglelefteq G$ is a cyclic subgroup for prime q, and $O_{q'}(G')$ is cyclic too. Also, the Sylow q-subgroup is the only non-cyclic Sylow subgroup of G.

Theorem 2.11. *Assume that G is one of the groups in the above theorems and S be a nilpotent non-normal subgroup of G. Then S is cyclic.*

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On the shape group of mapping spaces

T. Nasri*

Abstract

In 1974, V. L. Hansen computed the fundamental group of mapping space X^{S^1}, where X is connected. In this talk we obtain the fundamental group of mapping space $(X \times Y)^{S^1}$ for connected spaces X and Y. Also, in this talk we intend to the study of shape group of mapping spaces. In particular, using the result of Hansen we compute the shape group of mapping space X^{S^1}.

Keywords and phrases: Fundamental group, Shape group, Mapping space.

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1. Introduction

Given spaces X and Y, let X^Y denote the space of all continuous maps from X to Y wih the compact-open topology. The homotopy theory of function spaces first apears in the 1940s. papers computing homotopy groups of components of X^Y, for some spaces X and Y can be found in the literature since 1956. V. L. Hansen [1] computed the fundamental group of mapping space X^{S^1}, where X is connected. In this talk we obtain the fundamental group of mapping space $(X \times Y)^{S^1}$ for connected spaces X and Y. Let (X, x) be a pointed space and $p : (X, x) \longrightarrow (\mathbf{X}, \mathbf{x})$ is an Hpol$_*$-expansion of (X, x), then the shape homotopy group of (X, x) is denoted by $\check{\pi}_k(X, x)$ and defined as follows

$$\check{\pi}_k(X, x) = \lim_{\leftarrow} \pi_k(\mathbf{X}, \mathbf{x}).[2]$$

In this talk we intend to obtain the shape group of mapping spaces.

2. Main Results

For connected space X, let X^Y denote the space of continuous maps from Y to X equuipped with the compact-open topology. In particular, X^{S^1} denote the free loop space on X. Also, let ΩX denote the ordinary loop space on X. It is well-known that the map $p : X^{S^1} \longrightarrow X$ defined by $p(\omega) = \omega(1)$ is a fibration with fiber ΩX. We need the following result of Hansen.

Theorem 2.1. *[1,* Proposition 1*] Suppose that* $\pi_2(X, *) = 0$ *and let* $f : S^1 \longrightarrow X$ *be an arbitrary base point preserving map representing the homotopy class in* $\pi_1(X, *)$*. Then*

$$\pi_1(X^{S^1}, f) \cong C_{[f]}(\pi_1(X, *)).$$

Recall that the centralizer of an element g in a group G is the subgroup of G defined by $C_g(G) = \{g' \in G | gg' = g'g\}$. With the same argument of the above theorem we have the following result.

Theorem 2.2. *Let* $f : (S^1, 1) \longrightarrow (X, x_0)$ *and* $g : (S^1, 1) \longrightarrow (Y, y_0)$ *be two continuous pointed maps of connected spaces, Then*

$$\pi_1((X \times Y)^{S^1}, (f, g)) \cong C_{[(f,g)]}(\pi_1(X \times Y, (x_0, y_0))),$$

where $(f, g) : (S^1, 1) \longrightarrow (X \times Y, (x_0, y_0))$ *is given by* $(f, g)(z) = (f(z), g(z))$*.*

Corollary 2.3. *Let* $f : (S^1, 1) \longrightarrow (X, x_0)$ *and* $g : (S^1, 1) \longrightarrow (Y, y_0)$ *be two continuous pointed maps of connected spaces, Then*

$$C_{[(f,g)]}(\pi_1(X \times Y, (x_0, y_0))) \cong C_{[f]}(\pi_1(X, x_0)) \times C_{[g]}(\pi_1(Y, y_0)).$$

PROOF. We know that $(X \times Y)^{S^1} \approx X^{S^1} \times Y^{S^1}$. Then

$$\begin{aligned} C_{[(f,g)]}(\pi_1(X \times Y, (x_0, y_0))) &\cong \pi_1((X \times Y)^{S^1}, (f, g)) \\ &\cong \pi_1(X^{S^1} \times Y^{S^1}, (f, g)) \cong \pi_1(X^{S^1}, f) \times \pi_1(Y^{S^1}, g) \\ &\cong C_{[f]}(\pi_1(X, x_0)) \times C_{[g]}(\pi_1(Y, y_0)). \end{aligned}$$

□

In follow, we intend to obtain the shape group of mapping spaces. Let us recall from [2] that if $p : (X, x) \longrightarrow (\mathbf{X}, \mathbf{x})$ is an Hpol$_*$-expansion of (X, x), then the shape homotopy group of (X, x) is denoted by $\check{\pi}_k(X, x)$ and defined as follows

$$\check{\pi}_k(X, x) = \lim_{\leftarrow} \pi_k(\mathbf{X}, \mathbf{x}).$$

In particular if $(X, x) = \lim\limits_{\leftarrow}(X_i, x_i)$, where X_i is compact polyhedron for all $i \in I$, then

$$\check{\pi}_1(X, x) = \lim_{\leftarrow} \pi_1(X_i, x_i).[3, Remark 1]$$

Theorem 2.4. *Let* $(X, x) = \underleftarrow{\lim}(X_i, x_i)$, *where* X_i *is compact polyhedron for all* $i \in I$ *and* Y *be a compact metric space. Then* $(X, x)^{(Y,y)} \approx \underleftarrow{\lim}(X_i, x_i)^{(Y,y)}$

Theorem 2.5. *Let* $(X, x) = \underleftarrow{\lim}(X_i, x_i)$, *where* X_i *is compact polyhedron for all* $i \in I$ *and* Y *be a compact metric space. Then* $\check{\pi}_1((X, x)^{(Y,y)}) \cong \underleftarrow{\lim}\,\pi_1((X_i, x_i)^{(Y,y)})$.

Proof. Since Y is a compact metric and X_i is compact polyhedron for all $i \in I$, X_i^Y's are compact polyhedra and therefore $\underleftarrow{\lim}(X_i, x_i)^{(Y,y)}$ is an Hpol$_*$-expansion of $(X, x)^{(Y,y)}$. Thus by definition $\check{\pi}_1((X, x)^{(Y,y)}) \cong \underleftarrow{\lim}\,\pi_1((X_i, x_i)^{(Y,y)})$.

□

Corollary 2.6. *Let* $(X, x) = \underleftarrow{\lim}(X_i, x_i)$, *where* X_i *is connected compact polyhedron for all* $i \in I$. *Then*

$$\check{\pi}_1(X^{S^1}, f) = \underleftarrow{\lim}\, C_{[f]}\pi_1(X_i, x_i).$$

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Some Results Of Two-sided Group Graphs

Farzaneh Nowroozi Larki and Shahram Rayat Pisheh*

Abstract

In this paper, we study a family of graphs that can be considered the generalization of Cayley graphs and digraphs. For non-empty subsets L, R of group G, two-sided group digraph $\overrightarrow{2S}(G; L, R)$ has been defined as a digraph having the vertex set G, and an arc from x to y if and only if $y = l^{-1}xr$ for some $l \in L$ and $r \in R$.

Keywords and phrases: Cayley digraph, Cayley graph, Group. .

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1. Introduction

Let G be a finite group with identity element e and S be a subset of G such that $e \notin S$. The Cayley digraph on G with respect to S is defined as a digraph with vertex set G and an arc (x, y) (from vertex x to vertex y) if and only if $xy^{-1} \in S$ is denoted by $\overrightarrow{Cay}(G, S)$. The condition $e \notin S$ yields a digraph with no loop. Moreover, if $S = S^{-1}$ (where $S^{-1} = \left\{s^{-1} | s \in S\right\}$), then we have a simple undirected graph [2], which is called a Cayley graph, denoted by $Cay(G, S)$. In this definition, S can be considered an empty set, by which the related Cayley graph has no edge. It can be verified that the Cayley graph is connected if and only if S generates G ($G = \langle S \rangle$) [3]. Cayley graphs have many applications in different sciences such as biology, coding theory, computer science and interconnection networks. Cayley graphs were introduced by Arthur Cayley in 1878 [4]. In this paper, we study a generalization of Cayley digraphs introduced by Iradmusa and Praeger in 2016 [6]. Iradmusa and Praeger named it a two-sided group digraph (graph) denoted by $\overrightarrow{2S}(G; L, R)$ ($2S(G; L, R)$). They found conditions for the

* speaker

adjacency relation defining the digraphs to be symmetric, transitive or connected, etc. and they posed eight problems in their article [6].

Let G be a group and L, R are two non-empty subsets of G, then the two-sided group digraph $\overrightarrow{2S}(G; L, R)$ is defined with vertex set G and an arc (x, y) 'from x to y' if and only if $y = l^{-1}xr$ for some $l \in L$ and $r \in R$. The connection set of $\overrightarrow{2S}(G; L, R)$ is defined as the set $\hat{S}(L, R) = \{\lambda_{l,r} : l \in L, r \in R\}$, where $\lambda_{l,r}$ is a permutation of the form $\lambda_{l,r} : g \mapsto l^{-1}gr$, for certain $l, r \in G$. Note, if there are no loops and the adjacency relation is symmetric, then $\overrightarrow{2S}(G; L, R)$ will be regarded as a simple graph, and will be named a two-sided group graph. Let $x \in G$ be an arbitrary element; we define an equivalence relation on $L \times R$ as follows: $(l_1, r_1) \sim_x (l_2, r_2)$ if and only if $(x)\lambda_{l_1,r_1} = (x)\lambda_{l_2,r_2}$; then equivalence class including (l, r) is presented as $C_x(l, r) = \{(l', r')|(x)\lambda_{l',r'} = (x)\lambda_{l,r}, l' \in L, r' \in R\}$ and C_x is the set of all equivalence classes of $\sim_x$. It is obvious when $\Gamma = \overrightarrow{2S}(G; L, R)$ is an undirected graph, then valency(x) is equal to $|C_x|$. In other words, the *valency*(x) is corresponding to a partition of $|L||R|$.

Definition 1.1. *[6] Let G be a group with identity element e and two subsets L, R. Then a pair (L, R) has 2S-graph-property if both L and R are non-empty, and the following conditions hold:*
(1)$L^{-1}gR = LgR^{-1}$ for each $g \in G$;
(2)$L^g \cap R = \emptyset$ for each $g \in G$; and
(3)$(LL^{-1})^g \cap (RR^{-1}) = \{e\}$ for each $g \in G$.

For a vertex x of a two-sided group digraph $\overrightarrow{2S}(G; L, R)$, the arcs beginning with x, are the pair (x, y) with $y = (x)\lambda$, for some $\lambda \in \hat{S}(L, R)$, such elements y are called out-neighbours of x, and the number of distinct out-neighbours of x are called the out-valency of x. Similarly, the arcs ending in x are the pairs (y, x) with $(y)\lambda = x$, for some $\lambda \in \hat{S}(L, R)$, such elements y are called in-neighbours of x, and the number of distinct in-neighbours of x are called the in-valency of x. If there is a constant c such that each vertex x has out-valency c and in-valency c, then $\overrightarrow{2S}(G; L, R)$ is regular of valency c.

For any group G, the right regular representation and left regular representation give two regular subgroups of $Sym(G)$, each isomorphic to G, namely, $G_R = \{\lambda_{e,g}|g \in G\}$ and $G_L = \{\lambda_{g,e}|g \in G\}$.

Theorem 1.2. *[6] Let L, R be non-empty subsets of a group G, and $\Gamma = \overrightarrow{2S}(G; L, R)$.*
(1) $G_R \leq Aut(\Gamma)$ if and only if $L^{-1}gR = L^{-1}Rg$ for each $g \in G$; here $\Gamma = \overrightarrow{Cay}(G, L^{-1}R)$.
(2) $G_L \leq Aut(\Gamma)$ if and only if $L^{-1}gR = gL^{-1}R$ for each $g \in G$; here $\Gamma = \overrightarrow{Cay}(G, R^{-1}L)$.

Corollary 1.3. *Let L, R be non-empty subsets of group G, and $\Gamma = \overrightarrow{2S}(G; L, R)$. If $R \subseteq Z(G)$ or $L \subseteq Z(G)$, then $\Gamma = \overrightarrow{Cay}(G, L^{-1}R)$ or $\Gamma = \overrightarrow{Cay}(G, R^{-1}L)$, respectively. In particular, if L, R are inverse-closed subsets of G and $L, R \subseteq Z(G)$, then $\Gamma = \overrightarrow{Cay}(G, LR) = \overrightarrow{Cay}(G, RL)$.*

2. Main Results

Let G be a group with two non-empty subsets L, R. If l is an arbitrary element of L, so $l^{-1}lr = r$ and it implies (l, r) is an arc of $\Gamma = \overrightarrow{2S}(G; L, R)$, for each $l \in L, r \in R$. Similarly, it can be proved (r^{-1}, l^{-1}) is an arc as well. Thus, $\{l, r\}, \{r^{-1}, l^{-1}\}$ are edges of Γ, in the case Γ is undirected, so $valency(l) \geq |R|$ and $valency(r^{-1}) \geq |L|$. Hence, if Γ is an undirected regular graph, we will have $valency(x) \geq \frac{|L|+|R|}{2}$ for each $x \in G$.

Proposition 2.1. *Let G be a group with two non-empty subsets L, R and $\Gamma = 2S(G; L, R)$ is a regular, undirected graph, then $\frac{|L|+|R|}{2} \leq valency(x) \leq |L||R|$.*

Proposition 2.2. *Let L, R be non-empty subsets of group G, and $\Gamma = \overrightarrow{2S}(G; L, R)$.*
(1) If $|L| = 1$ (or $|R| = 1$) then Γ is a regular digraph of valency $|R|$ ($|L|$).
(2) If $L = \{l\}, R = \{r\}, l \neq r$, then (L, R) has 2S-graph-property if and only if:
$l^2 = r^2$, $l^2 \in Z(G)$ and $r \neq g^{-1}lg$ for each $g \in G$; in particular $l \neq r$.

Proof. *(1) Let $|L| = 1$, say $L = \{l\}$. If $l^{-1}xr_1 = l^{-1}xr_2$, then $r_1 = r_2$, so out-valency(x) = in-valency(x) = valency$(x) = |R|$, for each $x \in G$.*
(2) Suppose that $\Gamma = \overrightarrow{2S}(G; L, R)$, then according to the 2S-graph-property, we must have $l^{-1}gr = lgr^{-1}$ for each $g \in G$, thus $l^2g = gr^2$ for each $g \in G$, hence $l^2 = r^2$ and $l^2 \in Z(G)$. Also, the relation $g^{-1}Lg \cap R = \emptyset$ leads to $\{g^{-1}lg\} \cap \{r\} = \emptyset$, therefore $r \neq g^{-1}lg$, i.e. l, r are not in the same conjugacy class, specially $l \neq r$. It is clear, $(LL^{-1})^g \cap (RR^{-1}) = \{e\}$ is satisfied. The converse is obvious.

Theorem 2.3. *Let G be a group and L, R be non-empty subsets of G.*
(1) $\Gamma = \overrightarrow{2S}(G; L, R)$ is matching if and only if L and R are single-member having 2S-graph-property.
(2) If L, R are single-member subsets and pair (L, R) has the 2S-graph-property then the order of G is even.

Theorem 2.4. *Let G be a group and $L = \{l\}, R = \{r\}$ be single-member subsets of G and $n \geq 3$ is an integer number. Then digraph $\Gamma = \overrightarrow{2S}(G; L, R)$ has a cycle of length n (thus girth $\Gamma \leq n$) if and only if $l^n g = gr^n$ for some $g \in G$, and n is the least integer with this property.*

By Kuratowski's theorem [5] we know that a graph is planar if and only if it contains no subgraph that is a subdivision of either K_5 or $K_{3,3}$. Then we will have the next theorem.

Theorem 2.5. *Let L, R be non-empty subsets of a group G and let $\Gamma = 2S(G; L, R)$ be a two-sided group (undirected) graph. If we have $L \cap L^{-1}LR = \emptyset$, $R \cap L^{-1}RR = \emptyset$, $|L| \geq 3$ and $|R| \geq 3$, then Γ is non-planar.*

Theorem 2.6. *Let G be a group and L, R be subgroups of G, and $H = \{\lambda_{l,r} | l \in L, r \in R\}$. Then the group H acts on G, $\Gamma = \overrightarrow{2S}(G; L, R)$ is a graph with one loop on each vertex, $valency(x) = \frac{|H|}{|stab_H(x)|}$ for each $x \in G$ and $|H| = |L||R|$. In particular Γ is regular if and only if $valency(x) = |LR|$ for each $x \in G$. Graph Γ is connected if and only if $G = LR$ and otherwise, the number of connected components is equal to the number of double cosets of the pair (L, R).*

Corollary 2.7. *Let G be a group and L, R be subgroups of G, $\Gamma = \overrightarrow{2S}(G; L, R)$. Then domination number of Γ is the number of double cosets of the pair (L, R).*

Theorem 2.8. *Let L, R be non-trivial subgroups of a group G.*
(1) $\Gamma = \overrightarrow{2S}(G; L, R)$ is a regular graph with one loop on each vertex, of valency strictly less than $|L|\,|R|$, if and only if $|L \cap R| > 1$.
(2)The valency of e in graph $\Gamma = \overrightarrow{2S}(G; L, R)$ is one, if and only if $L = R = \{e\}$.

Theorem 2.9. *Let G be a group, and $|G| = p^{\alpha}q^{\beta}m$, where p and q are distinct prime numbers and $\gcd(m, p) = 1, \gcd(m, q) = 1$. Let L and R be p-Sylow subgroup and q-Sylow subgroup of G, respectively. Suppose that $L^{\#} = L - \{e\}, R^{\#} = R - \{e\}$ and $\Gamma = \overrightarrow{2S}(G; L^{\#}, R^{\#})$, then pair $(L^{\#}, R^{\#})$ has the 2S-graph-property, therefore Γ is a simple graph and it is regular of valency $(p^{\alpha} - 1)(q^{\beta} - 1)$.*

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n-tensor degree of finite groups

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Abstract

In the present paper we talk about n-tensor degree of finite group G. It is the probability of randomly chosen elements x and y of the group G with $x^n \otimes y = 1_\otimes$ in the non-abelian tensor square $G \otimes G$ and denoted by $\mathcal{P}_n^{\otimes}(G)$ for any positive integer n

Keywords and phrases: Non-abelian tensor product, tensor centre, tensor degree .

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1. Introduction

Let G and H be two groups. The non-abelian tensor product of the groups G and H, which is denoted by $G \otimes H$, was introduced by R. Brown et. al. [1–3], and it is generated by the symbols $g \otimes h$. The groups G and H must act on each other and by conjugations on themselves on the left and satisfy the following relations:

$$g_1 g_2 \otimes h = ({}^{g_1}g_2 \otimes {}^{g_1} h)(g_1 \otimes h) \quad \text{and} \quad g \otimes h_1 h_2 = (g \otimes h_1)({}^{h_1}g \otimes {}^{h_1} h_2).$$

All these actions must be compatible in the sense that

$${}^{({}^{g_1}h)}g_2 = {}^{g_1}({}^{h}({}^{g_1^{-1}}g_2)), \ {}^{({}^{h_1}g)}h_2 = {}^{h_1}({}^{g}({}^{h_1^{-1}}h_2)),$$

for all $g, g_1, g_2 \in G$ and $h, h_1, h_2 \in H$. In particular, $G \otimes G$ is the tensor square of the group G.

The notion of tensor centre of a group G introduced by G. Ellis [4] as follows:

$$Z^{\otimes}(G) = \{g \in G \mid g \otimes x = 1_\otimes, \ \forall x \in G\},$$

* speaker

in which $1_\otimes$ is the identity element of $G \otimes G$.

Also the set

$$C_G^\otimes(x) = \{y \in G : x \otimes y = 1_\otimes\},$$

is the *tensor centralizer* of the element x in G. One can easily see that $Z^\otimes(G) = \cap_{x\in G} C_G^\otimes(x)$.

We denote $\mathcal{P}_n^\otimes(G)$ to be the probability that for any two randomly chosen elements x and y in the finite group G, such that $x^n \otimes y = 1_\otimes$. In fact

$$\mathcal{P}_n^\otimes(G) = \frac{|\{(x, y) \in G \times G : x^n \otimes y = 1_\otimes\}|}{|G|^2}.$$

It is easily seen that

$$\mathcal{P}_n^\otimes(G) = \frac{1}{|G|} \sum_{x\in G} |C_G^\otimes(x^n)|.$$

Clearly for $n = 1$, we obtain *tensor degree* of G (see [5] for more information).

If H is a subgroup of a finite group G, we introduce the following notion

$$\mathcal{P}_n^\otimes(H, G) = \frac{|\{(h, g) \in H \times G : h^n \otimes g = 1_\otimes\}|}{|H||G|},$$

which is called *relative n-tensor degree* of G, with respect to the subgroup H.

Clearly, if $H = G$ then $\mathcal{P}_n^\otimes(G) = \mathcal{P}_n^\otimes(G, G)$. Note that, if G is a group with $Z^\otimes(G) = G$ or G has exponent dividing n, then both $\mathcal{P}_n^\otimes(H, G)$ and $\mathcal{P}_n^\otimes(G)$ are equal to one.

In the following result, we compare $\mathcal{P}_n^\otimes(H, G)$ with $\mathcal{P}_n^\otimes(H)$.

Lemma 1.1. *Let H be a subgroup of a finite group G. Then $\mathcal{P}_n^\otimes(H, G) \le \mathcal{P}_n^\otimes(H)$, for all natural number n.*

In particular, the equality holds when $G = HZ^\otimes(G)$.

Lemma 1.2. *Let H be a proper subgroup of a given finite group G. Then for every natural number n*

$$|H|\mathcal{P}_n^\otimes(H, G) \le |G|\mathcal{P}_n^\otimes(G).$$

Using a similar argument as in the proof of the above lemma, we have the following

Lemma 1.3. *Let H and K be subgroups of a finite group G with K is contained in H. Then*

$$\frac{1}{[G:K]}\mathcal{P}_n^\otimes(K, H) \le \frac{1}{[H:K]}\mathcal{P}_n^\otimes(K, G) \le \mathcal{P}_n^\otimes(H, G).$$

The following result compares n-tensor degree of a given group G with its factor groups.

Lemma 1.4. *Let N be a normal subgroup of a finite group G. Then $\mathcal{P}_n^\otimes(G) \le \mathcal{P}_n^\otimes(G/N)$, for every natural number n.*

In particular, $\mathcal{P}_n^\otimes(G) = \mathcal{P}_n^\otimes(G/N)$, when $N \le Z^\otimes(G)$.

The next interesting result follows easily from the definition of n-tensor degree and can be extended to a finite number of groups.

Proposition 1.5. *Let G_1 and G_2 be two groups with coprime orders. Then* $\mathcal{P}_n^{\otimes}(G_1 \times G_2) = \mathcal{P}_n^{\otimes}(G_1) \times \mathcal{P}_n^{\otimes}(G_2)$.

2. Main Results

Here, considering the previous discussion, we investigate some properties of n-tensor degree of finite groups.

The following upper bound is useful in proving our main results.

Theorem 2.1. *Let G be a group with $G/Z^{\otimes}(G)$ elementary abelian p-group of rank r, for some prime number p. Then $\mathcal{P}_n^{\otimes}(G) = 1$, when p divides n. Otherwise, $\mathcal{P}_n^{\otimes}(G) \leq \frac{p^r+p-1}{p^{r+1}}$.*

The following example confirms the above theorem.

Example 2.2. *Consider the metacyclic group*

$$G = \langle x, y : x^9 = y^3, yxy^{-1} = x^4 \rangle,$$

then $G/Z^{\otimes}(G) \cong C_3 \times C_3$ and by Theorem 2.1 if 3 divides n, we have $\mathcal{P}_n^{\otimes}(G) = 1$, otherwise $\mathcal{P}_n^{\otimes}(G) \leq \frac{11}{27}$. Now, by using GAP [6] we obtain

$$\mathcal{P}_1^{\otimes}(G) = \mathcal{P}_2^{\otimes}(G) = \mathcal{P}_4^{\otimes}(G) = \mathcal{P}_5^{\otimes}(G) = \mathcal{P}_7^{\otimes}(G) = \mathcal{P}_8^{\otimes}(G) = \mathcal{P}_{10}^{\otimes}(G) = \frac{17}{81},$$

$$\mathcal{P}_3^{\otimes}(G) = \mathcal{P}_6^{\otimes}(G) = \mathcal{P}_9^{\otimes}(G) = 1.$$

Using the same argument we have the following

Lemma 2.3. *Let H be a subgroup of a finite group G such that $\frac{H}{H \cap Z^{\otimes}(G)}$ is elementary abelian p-group of rank r, for some prime p.*
(i) If p divides n, then $\mathcal{P}_n^{\otimes}(H, G) = 1$.
(ii) If p does not divide n and $Z^{\otimes}(G) = Z^{\otimes}(H)$, then

$$\mathcal{P}_n^{\otimes}(H, G) \leq \frac{p^r + p - 1}{p^{r+1}}.$$

The following theorem gives the exact value for the probability $\mathcal{P}_n^{\otimes}(G)$, when G is elementary abelian p-group.

Theorem 2.4. *Let G be elementary abelian p-group of rank r. If p divides n, then $\mathcal{P}_n^{\otimes}(G) = 1$ otherwise $\mathcal{P}_n^{\otimes}(G) = \frac{2p^r-1}{p^{2r}}$.*

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Some results in Q_1-groups

Mozhgan Rezakhanlou

Abstract

A finite group G is called a $\mathbb{Q}_1$-group if all of its non-linear irreducible characters are rational valued. In this paper we will find the general stracture of a metabelian $\mathbb{Q}_1$-group.

Keywords and phrases: $\mathbb{Q}_1$-group, rational group, metabelian.

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1. Introduction

Several mathematicians have studied rational groups from various aspects. Some recent works about classification of these groups can be found in [6], [10] and [11]. At the beginning of this research, we recall two main definitions. In the next two sections, we introduce some results about classification of these groups.

Definition 1.1. *A finite group G is called a rational group or a Q-group,if all its irreducible characters are rational valued.*
A detaild discution of the structure of Q-groups can be found in [3],[7], and [8]. Examples of Q-groups are the symmetric groups S_n.

Theorem 1.2. *G is a Q-group if and only if every $x \in G$ is conjugate to x^m, where $m \in \mathbb{N}$, $(o(x), m) = 1$ and this means that for every $x \in G$: $\dfrac{N_G(\langle x \rangle)}{C_G(\langle x \rangle)} \simeq \mathrm{A}ut(\langle x \rangle)$ [8].*

Definition 1.3. *A finite group G is called a Q_1-group if all of its non-linear irreducible characters are rational valued.*

The notion of Q_1-group was first introduced in [3]. Examples of Q_1-groups are Abelian groups. A_4 is a Q_1-group which is not a Q-group (that called a $Q_1^{'}$-group). In [5], Darafsheh et al obtained some helpful results about the structure of Q_1-groups. Some of these results are as follows:

Theorem 1.4. *[5] Let G be a Q_1-group. If N is a normal subgroup of G, then G/N is a Q_1-group.*

Theorem 1.5. *[5] Let G be a Q_1-group, then $|G|$ is even and $Z(G)$ is an elementary Abelian 2-group.*

Theorem 1.6. *[5] Let G be a nonsolvable group, then G is Q_1-group if and only if G is a Q-group.*

Theorem 1.7. *Let G be a non-Abelian finite group, then G is Q_1-group if and only if every element of $V(G) = \langle g \in G | \exists \chi \in nl(G) : \chi(g) \neq 0 \rangle$ is a rational element.*

Note that $V(G)$ is called vanishing-off subgroup and it is helpful for classification of some Q_1-groups [5]. It is clear that every $\mathbb{Q}$-group is a $\mathbb{Q}_1$-group. The elementary properties of $\mathbb{Q}_1$-groups can be found in [5].

Throughout the paper we consider finite solvable groups, and we employ the following notation and terminology:

The semi-direct product of a group K with a group H is denoted by $K : H$. The symbol $\mathbb{Z}_n$ denotes a cyclic group of order n. For a prime p and a non-negative integer n, the symbol $E(p^n)$ denotes the elementary Abelian p-group of order p^n.

We introduce some more notation. Let G be a finite group. Let $nl(G)$ denote the set of non-linear irreducible characters of G.

An element $x \in G$ is called rational if $\chi(x) \in \mathbb{Q}$ for every $\chi \in Irr(G)$, otherwise it is called an irrational element. Also, $\chi \in Irr(G)$ is called a rational character if $\chi(x) \in \mathbb{Q}$ for every $x \in G$.

2. Classification of Q-groups

The complete classification of Q-groups has not been done yet. But some special Q-groups have been classified.

Definition 2.1. *A finite group G is a Frobenius group if it contains a proper subgroup $H \neq \{1\}$ called a Frobenius complement such that $H \cap H^x = \{1\}$ for all $x \notin H$.*

By Frobenius Theorem, a Frobenius group G with complement H, has a normal subgroup K, called Frobenius kernel such that $H \cap K = 1$, $G = HK$, (G=K:H) for example S_3 and A_4 are frobenius groups.

Theorem 2.2. *[6] If G is a Frobenius Q-group, then exactly one of the following occurs:*
(1) $G \cong E(3^n) : \mathbb{Z}_2$,where $n \geqq 1$ and $\mathbb{Z}_2$ acts on $E(3^n)$ by inverting every non-identity element.
(2) $G \cong E(3^{2m}) : \mathbb{Q}_8$, where $m \geqq 1$ and $E(3^{2m})$ is a direct sum of m copies of the 2-dimensional irreducible representation of $\mathbb{Q}_8$ over the field with 3 elements.
(3) $G \cong E(5^2) : \mathbb{Q}_8$, where $E(5^2)$ is the 2-dimensional representation of $\mathbb{Q}_8$ over the field with 5 elements.

Definition 2.3. *A finite group G is called a 2-Frobenius group if it has a normal series $1 \lhd H \lhd K \lhd G$, where G/H and K are Frobenius groups with kernels K/H and H , respectively.*

For example, S_4 is a 2-Frobenius group. Note that 2-Frobenius groups are always solvable.

Theorem 2.4. *[4] Let G be a 2-Frobenius Q-group. Then, we have:*
(1) $|G| = 2^n.3$ where n is an odd integer.
(2) G has a normal 2-subgroup N such that $\frac{G}{N} \cong S_4$.
(3) If H is a minimal normal subgroup of G then $G \cong S_4$.

Theorem 2.5. *[1] A rational group G has only one non-linear irreducible character if and only if G is isomorphic either to the symmetric group on three letters S_3, to the dihedral group D_4 of order 8, or to the quaternion group Q_8 of order 8.*

3. Classification of Q_1-groups

Theorem 3.1. *[10] If G is a Frobenius Q_1-group, then one of the following occures:*
(1) $G \cong E(p^n) : \mathbb{Z}_t$, where p is an odd prime, $n \geq 1$ and $t \geq 1$ is even.
(2) $G \cong G' : \mathbb{Z}_t$, where the derived subgroup G' of G is a rational 2-group and $t \geq 1$ is odd.
(3) $G \cong E(5^2) : \mathbb{Q}_8$ or $G \cong E(3^{2m}) : \mathbb{Q}_8$, where $m \geq 1$.
(4) $G \cong E(p^n) : H$, where p is a Fermat prime, $n \geq 1$ and H is a metacyclic group of order $2^m q$, for some Fermat prime q and $m \geq 1$.

The groups described in (1) are not necessarily Q_1-groups, and, in (2), the derived group G' is not necessarily elementary Abelian. By [5], all groups of the form (3) are Q_1-groups. We recall that a Fermat prime is a prime of the form $2^{2^t} + 1$ for some $t \geq 0$.

Lemma 3.2. *[10] Let $G = K : H$ be a Frobenius Q_1-group. If H is a Q-group, then G is a Q-group. If H is Abelian, then $K = G' = V(G)$, and thus K is a Q-group.*

Proposition 3.3. *[10] Suppose that $G = K : H$ is a Frobenius group such that H is Abelian of odd order. Then G is a Q_1-group if and only if K is a Q-group.*

Theorem 3.4. *Let $G = H : K$ be a Frobenius Q_1-group with non-Abelian complement. If $G_2 \in Syl_2(G)$ and G_2 is isomorphic to a generalized quaternion group. Then G is a Q-group [10].*

In [11], the research is based on whether a Sylow 2-subgroup of a Q_1-group G is included in $V(G)$ or not.

Theorem 3.5. *[11] Suppose that G is non-Abelian solvable Q_1-group with Sylow 2-subgroup P. Then one of the following occures:*

(1) If $P \subseteq V(G)$ then $G \simeq V(G) : \mathbb{Z}_m$ or $G \simeq V(G) : E(p^n)$, where m is an odd integer and p is coprime to $|V(G)|$.

(2) If P is non-Abelian and $P \nsubseteq V(G)$, then $G \simeq K : P$, where K is a $\{3, 5, 7\}$-group.

(3) If P is Abelian and $P \nsubseteq V(G)$ then $G \simeq G' : (\mathbb{Z}_m \times E(2^n))$, where the derived subgroup G' is a Hall subgroup of odd order and $m, n \in \mathbb{Z}$.

Theorem 3.6. *[11] Suppose that G is a solvable Q_1-group and P is a Sylow 2-subgroup of G. If P has nilpotence class two, then*

(a) If $P \nsubseteq V(G)$ then P is a Q_1-group.

(b) If $P \subseteq V(G)$ then P is a Q-group.

Definition 3.7. *A finite group G is called a metabelian group if G has a normal subgroup N such that N and G/N are Abelian groups.*

For example, S_3 is a metabelian group.

Theorem 3.8. *[2] Let G be a finite metabelian group with all nonlinear irreducible characters rational. Then the expotent of the commutator group G' is a prime or divides 16, 24, or 40. If G' is also cyclic, then its order is a prime or divides 12.*

In a new research we classified metabelian Q_1-groups. For this classifications we used some theorems in [11] and [2]. The results of these classifications are as follows [9]:

Proposition 3.9. *Let G be a metabelian $\mathbb{Q}_1$-group and $P \in Syl_2(G)$. If $P \subseteq V(G)$ then $G \cong (E(3^n) : P) : \mathbb{Z}_m$ or $G \cong P : \mathbb{Z}_m$, where m is a positive integer that is coprime to 6.*

Proposition 3.10. *Suppose that G is a metabelian $\mathbb{Q}_1$-group and let $P \in Syl_2(G)$. If $P \nsubseteq V(G)$ and P is non-abelian group, then one of the following occures:*

(1) *G is a 2-group and $exp(G')$ devides 16.*

(2) *$G \cong E(3^n) : P$ or $G \cong E(5^n) : P$. Also $exp(P')$ devides 8.*

Proposition 3.11. *Let G be a metabelian $\mathbb{Q}_1$-group and $P \in Syl_2(G)$. If $P \nsubseteq V(G)$ and P is abelian then $G \cong E(p^n) : ((\mathbb{Z}_m) \times E(2^n))$, where p is an odd prime.*

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On non-vanishing elements in finite groups

S. M. Robati

Abstract

Let G be a finite group. We denote by Van(G) the set of vanishing elements of G, in which an element g in G is vanishing element if there exists some irreducible character χ of G such that $\chi(g) = 0$. In this paper, we study some influences the structure of $G - Van(G)$ on the structure of G.

Keywords and phrases: Finite groups, vanishing elements, conjugacy classes..

2010 *Mathematics subject classification:* Primary: 20C15, 20E45.

1. Introduction

Let G be a finite group. An element g in G is vanishing element if there exists some irreducible character χ of G such that $\chi(g) = 0$. The vanishing set, Van(G), denotes the set of all vanishing elements of G. A result of Burnside [2] states that if χ is a non-linear irreducible character, then $\chi(g) = 0$ for some $g \in G$. Thus G is a non-abelian group if and only if $Van(G) \neq \emptyset$.

There are many paper on investigating some influences the structure of $Van(G)$ on the algebraic structure of G. In [1], Bubboloni et al. classified the groups whose vanishing elements are involutions. Additionally, in [5] we find the structure of groups whose vanishing elements are of odd order.

On the other hand, there are some results concerning the influence of non-vanishing elements of G on the group structure G, which $g \in G$ is a non-vanishing element if $\chi(g) \neq 0$ for all irreducible characters χ of G. It is clear that the set of non-vanishing elements is $G - Van(G)$. A well-known conjecture in [3] assert that $G - Van(G)$ is a subset of the Fitting subgroup $F(G)$ for solvable group G. In [3], authors proved that if G is a solvable group and $x \in G$ of odd order, then $x \in F(G)$. Furthermore, Miyamoto in [4] proved that if G is solvable, then $Van(G) \neq G$.

Moreover, an area of research in group theory considers the relationship between conjugacy class sizes of G and the structure of G. For instance, the S_3-conjecture, which states that S_3 is the only non-abelian finite group with conjugacy classes of distinct sizes, is an open conjecture solved for solvable groups in [7]. On the other hand, in [6] we find the structure of Frobenius groups with at most two conjugacy classes of each size.

In this paper, we show a connection between the size of vanishing conjugacy classes of G (conjugacy classes of G contained in $G - Van(G)$) and the structure of group G.

2. Main Results

Lemma 2.1. *Let G be a finite group. If $cl(a)$ is a conjugacy class of G contained in $G - Van(G)$, then $cl(az)$ is a subset of $G - Van(G)$ for all $z \in Z(G)$.*

Proof. We have $\chi(a) \neq 0$ for all $\chi \in Irr(G)$. By Problem 3.12 of [2], we can write that

$$\begin{aligned}\chi(a)\chi(z) &= \frac{\chi(1)}{|G|}\sum_{h\in G}\chi(az^h)\\ &= \frac{\chi(1)}{|G|}\sum_{h\in G}\chi(az)\\ &= \chi(1)\chi(az)\end{aligned}$$

for all $z \in Z(G)$. Therefore $\chi(az) \neq 0$ for all $\chi \in Irr(G)$.

□

Theorem 2.2. *Let G be a finite group. If non-central conjugacy classes of G contained in $G - Van(G)$ are of distinct sizes, then $Z(G)$ is trivial.*

Proof. Assume that z is an element of $Z(G)$ and a is a non-central non-vanishing element of G. By Lemma 2.1, we observe that $cl(a)$ and $cl(az)$ are two conjugacy classes of the same size contained in $G - Van(G)$, and so by the hypothesis we have $cl(a) = cl(az)$. By Problem 3.12 of [2], we can see that

$$\begin{aligned}\chi(a)\chi(z) &= \frac{\chi(1)}{|G|}\sum_{h\in G}\chi(az^h)\\ &= \frac{\chi(1)}{|G|}\sum_{h\in G}\chi(az) = \frac{\chi(1)}{|G|}\sum_{h\in G}\chi(a)\\ &= \chi(1)\chi(a).\end{aligned}$$

Therefore since $\chi(a) \neq 0$, then $\chi(1) = \chi(z)$ for all $\chi \in Irr(G)$. Hence

$$z \in \bigcap_{\chi\in Irr(G)} ker\chi = \{1\}$$

and so $Z(G) = \{1\}$.

□

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The Complexity of Commuting Graphs and Related Topics

F. Salehzadeh* and A. R. Moghaddamfar

Abstract

Let G be a group and associate with G a graph $C(G)$ as follows: the vertices are the elements of G and two vertices x, y are joined by an edge if and only if x and y commute as elements of G. This graph, which is always connected, is called commuting graph of G. In this talk, we will provide some formulas concerning the complexities of commuting graphs associated with certain finite groups. We will also classify all nonabelian groups, up to isomorphism, which have an n-abelian partition for $n = 2, 3$.

Keywords and phrases: Commuting graph, complexity, n-abelian partition, AC-group.

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1. Introduction

Throughout this talk all graphs are finite, simple (i.e. without loops and multiple edges) and undirected. Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a graph. We say that $I \subseteq V_\Gamma$ is independent if and only if for all $u, v \in I$, $uv \notin E_\Gamma$, and $C \subseteq V_\Gamma$ is a clique if and only if for all $u, v \in C$ with $u \neq v$, $uv \in E_\Gamma$. The subsets $V_1, V_2, \ldots, V_k$ of V_Γ form a partition of V_Γ if and only if for all $i, j \in [1, k]$ with $i \neq j$, $V_i \cap V_j = \emptyset$ and $V = V_1 \cup V_2 \cup \cdots \cup V_k$. A partition of V_Γ into independent sets $I_1, \ldots, I_m$ and cliques $C_1, \ldots, C_n$, is a (m, n)-partition of Γ (see [2]):

$$V_\Gamma = I_1 \uplus I_2 \uplus \cdots \uplus I_m \uplus C_1 \uplus C_2 \uplus \cdots \uplus C_n.$$

Note that $(1, 1)$-partitionable graphs are called split graphs (see [5]), $(1, 0)$-partitionable graphs are called edgeless graphs, $(0, 1)$-partitionable graphs are called complete graphs. In particular,

* speaker

in the case when $m = 0$ or $n = 0$, we essentially split Γ into n cliques, $V_\Gamma = C_1 \uplus C_2 \uplus \cdots \uplus C_n$, or m independent sets, $V_\Gamma = I_1 \uplus I_2 \uplus \cdots \uplus I_m$, respectively.

We now concentrate on a graph associated with a finite group, which is called commuting graph. Let G be a finite group and X a non-empty subset of G. The *commuting graph* $C(X)$, has X as its vertex set with two distinct elements of X joined by an edge when they *commute* as two elements of G. We say that $X \subseteq G$ is a *commuting set* if and only if for all $x, y \in X$, $xy = yx$. Clearly, X is a commuting subset of G if and only if $C(X)$ is complete. Commuting graphs have been investigated by many authors in various contexts, for example see [3, 6]. We notice that when $1 \in X$, $C(X)$ is connected, so one can talk about the number of spanning trees (or *complexity*) of this graph, which is denoted by $\kappa(X)$. Especially, in the case when X is a commuting subset of G, by Cayley's formula we obtain $\kappa(X) = |X|^{|X|-2}$. In the sequel, we have provided some formulas concerning the complexities of commuting graphs associated with certain finite groups (see [7]). It is clear that $C(G)$ is $(0, n)$-partitionable if and only if G can be partitioned into n commuting subsets. This suggests the following definition.

Definition 1.1. Let G be a nonabelian group, $A \subseteq G$ an abelian subgroup and $n \geqslant 2$ an integer. We say that G has an *n-abelian partition with respect to A*, if there exists a partition of G into A and n disjoint commuting subsets $A_1, A_2, \ldots, A_n$ of G, $G = A \uplus A_1 \uplus A_2 \uplus \cdots \uplus A_n$, such that $|A_i| > 1$ for each $i = 1, 2, \ldots, n$.

Note that, the condition $n \geqslant 2$ in Definition 1.1 is needed. Indeed, if $n = 1$, then $G = A \uplus A_1$, and so $G = \langle A_1 \rangle$. Furthermore, since A_1 is a commuting set, this would imply G is abelian, which is not the case. Also, if $Z = Z(G)$, $|Z| \geqslant 2$, $|G : Z| = 1 + n \geqslant 4$ and $T = \{x_0, x_1 \ldots, x_n\}$ is a transversal for Z in G, where $x_0 \in Z$, then, as the cosets of the center are abelian subsets of G we have the following n-abelian partition for G with respect to Z: $G = Z \uplus Zx_1 \uplus \cdots \uplus Zx_n$. However, there are centerless groups G for which there is no n-abelian partition with respect to an abelian subgroup, for every n. For example, consider the symmetric group $\mathbb{S}_3$ on 3 letters.

Another purpose of this talk is to classify, up to isomorphism, all groups G which have an n-abelian partition for $n = 2, 3$ (see [7]).

2. Main Results

Let G be a nonabelian group and $A \subset G$ an abelian subgroup. If G has an n-abelian partition with respect to A, $G = A \uplus A_1 \uplus \cdots \uplus A_n$, then $C(A) = K_{|A|}$, and $C(A_i \cup \{1\}) = K_{|A_i|+1}$, $i = 1, 2, \ldots, n$. Hence, by [6, Corollary 2.7], we get

$$\kappa(G) \geqslant |A|^{|A|-2} \prod_{i=1}^{n} (|A_i| + 1)^{|A_i|-1}.$$

Especially, if $Z = Z(G)$ the center of the group G is nontrivial, then we have the n-abelian partition of G; $G = Z \uplus Zx_1 \uplus \cdots \uplus Zx_n$; and so

$$\kappa(G) \geqslant |Z|^{|Z|-2}(|Z|+1)^{(|Z|-1)n}.$$

A *noncommuting set* of a group G (i.e., an independent set in commuting graph $\mathcal{C}(G)$) has the property that no two of its elements commute under the group operation. We denote by $\mathrm{nc}(G)$ the maximum cardinality of any noncommuting set of G (the independence number of $\mathcal{C}(G)$). Denote by $\mathrm{k}(G)$ the number of distinct conjugacy classes of G. If G has an n-abelian partition, then the pigeon-hole principle gives $\mathrm{nc}(G) \leqslant n+1$. Thus, by Corollary 2.2 (a) in [1], we obtain

$$|G| \leqslant \mathrm{nc}(G) \cdot \mathrm{k}(G) \leqslant (n+1)\mathrm{k}(G),$$

which immediately implies that

$$n \geqslant \left\lfloor \frac{|G|}{\mathrm{k}(G)} \right\rfloor - 1. \tag{1}$$

Therefore, we have found a lower bound for n when $\mathrm{k}(G)$ is known.

Some Examples. Let $G = L_2(q)$, where $q \geqslant 4$ is a power of 2. We know that $|G| = q(q^2-1)$ and $\mathrm{k}(G) = q+1$. Thus, if G has an n-abelian partition, then by Eq.(1), we get $n \geqslant q^2 - q - 1$. In particular, since $\mathbb{A}_5 \cong L_2(4)$, if $\mathbb{A}_5$ has an n-abelian partition, then $n \geqslant 11$. In fact, $\mathbb{A}_5$ has a 20-abelian partition, as follows:

$$\mathbb{A}_5 = A \uplus A_1^{\#} \uplus A_2^{\#} \uplus \cdots \uplus A_{20}^{\#},$$

where $A_i^{\#} = A_i \setminus \{1\}$, for every i, and $A, A_1, \ldots, A_5$ are Sylow 5-subgroups of order 5, $A_6, A_7, \ldots, A_{15}$ are Sylow 3-subgroups of order 3, $A_{16}, A_{17}, \ldots, A_{20}$ are Sylow 2-subgroups of order 4.

Similarly, if $G_1 = \mathrm{GL}(2,q)$ and $G_2 = \mathrm{GL}(3,q)$, q a prime power, then we have $|G_1| = (q^2-q)(q^2-1)$ and $\mathrm{k}(G_1) = q^2-1$, while $|G_2| = (q^3-1)(q^3-q)(q^3-q^2)$ and $\mathrm{k}(G_2) = q^3-q$. Again, if G_i has an n_i-abelian partition, for $i = 1, 2$, by Eq. (1), we obtain $n_1 \geqslant q(q-1)-1$ and $n_2 \geqslant q^2(q^3-1)(q-1)-1$.

Lemma 2.1. ([7]) *Let a graph Γ with n vertices contain $m < n$ universal vertices. Then $\kappa(\Gamma)$ is divisible by n^{m-1}.*

Corollary 2.2. *Let G be a nonabelian group of order n with the center of order m. Then $\kappa(G)$ is divisible by n^{m-1}.*

2.1. Computing some explicit formulas for $\kappa(G)$ In what follows, we consider the problem of finding the complexity of commuting graphs associated with certain finite groups.

As the first result, we shall give an explicit formula for $\kappa(L_2(2^n))$.

Theorem 2.3. ([7]) *Let $q = 2^n$, where $n \geqslant 2$ is a natural number. Then there holds*

$$\kappa(L_2(q)) = q^{(q-2)(q+1)}(q-1)^{(q-3)q(q+1)/2}(q+1)^{(q-1)^2q/2}.$$

In particular, if $q = 4$, then $L_2(4) \cong L_2(5) \cong \mathbb{A}_5$ and so $\kappa(\mathbb{A}_5) = 2^{20} \cdot 3^{10} \cdot 5^{18}$.

In the next result, we will concentrate on nonabelian groups G in which the centralizer of every noncentral element of G is abelian. Such groups are called *AC-groups*. The smallest nonabelian AC-group is $\mathbb{S}_3$. There are also many infinite families of AC-groups, such as:

- Dihedral groups D_{2k} ($k \geqslant 3$), defined by

$$D_{2k} = \langle x, y \mid x^k = y^2 = 1, yxy^{-1} = x^{-1}\rangle.$$

- Semidihedral groups SD_{2^k} ($k \geqslant 4$), defined by

$$SD_{2^k} = \langle x, y \mid x^{2^{k-1}} = y^2 = 1, yxy^{-1} = x^{-1+2^{k-2}}\rangle.$$

- Generalized quaternion groups Q_{4k} ($k \geqslant 2$), defined by

$$Q_{4k} = \langle x, y \mid x^{2k} = 1, y^2 = x^k, yxy^{-1} = x^{-1}\rangle.$$

- Simple groups $L_2(2^k)$ ($k \geqslant 2$), and general linear groups $\mathrm{GL}(2, q)$, $q = p^k > 2$, p a prime.

Theorem 2.4. *Let G be a finite nonabelian AC-group of order n with center of order m. Let $C_G(x_1), C_G(x_2), \ldots, C_G(x_t)$ be all distinct centralizers of noncentral elements of G and $m_i = |C_G(x_i) \setminus Z(G)|$, for $i = 1, 2, \ldots, t$. Then, there holds*

$$\kappa(G) = n^{m-1}m^{t-1}\prod_{i=1}^{t}(m_i + m)^{m_i-1}.$$

In particular, if G is a centerless AC-group, then we have

$$\kappa(G) = \prod_{i=1}^{t}(m_i + 1)^{m_i-1} = \prod_{i=1}^{t}|C_G(x_i)|^{|C_G(x_i)|-2}.$$

Theorem 2.4, together with some rather technical computations (see [4]) yields some special results which are summarized in Table 1.

Table 1. $\kappa(G)$ for some special AC-groups G.

G		n	m	t	m_i	$\kappa(G)$
D_{2k}	k odd	$2k$	1	$k+1$	$k-1, 1, \ldots, 1$	k^{k-2}
D_{2k}	k even	$2k$	2	$k/2+1$	$k-2, 2, \ldots, 2$	$2^{\frac{3k+2}{2}}k^{k-2}$
Q_{4k}	$k \geqslant 2$	$4k$	2	$k+1$	$2k-2, 2, \ldots, 2$	$2^{5k-1}k^{2k-2}$
SD_{2^k}	$k \geqslant 4$	2^k	2	$2^{k-1}+1$	$2^{k-1}-2, 2, \ldots, 2$	$2^{(2^{k-2}-1)(2k+1)+4}$
P	p prime	p^3	p	$p+1$	$p^2-p, \ldots, p^2-p$	p^{2p^3-5}

2.2. Groups having an n-abelian partition

Theorem 2.5. ([7]) *The following conditions on a nonabelian group G are equivalent:*

(1) *G has a 2-abelian partition with respect to an abelian subgroup A.*

(2) *$G = P \times Q$, where $P \in \mathrm{Syl}_2(G)$ with $P/Z(P) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and Q is abelian, and $A = \langle Z(G), t\rangle$, where t is an involution outside of $Z(G)$.*

Corollary 2.6. *Let G be a group having a 2-abelian partition and let $Z(G)$ be its center of order m. Then, we have $\kappa(G) = 2^{5m-5}m^{4m-2}$.*

Theorem 2.7. ([7]) *The following conditions on a nonabelian group G are equivalent:*

(1) *G has a 3-abelian partition with respect to an abelian subgroup A.*

(2) *$|Z(G)| \geqslant 2$ and $G/Z(G)$ is isomorphic to one of the following groups: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{S}_3$. In the first case, $A = Z(G)$, and in two other cases $A = \langle Z(G), x\rangle$, where x is an element of order 3 outside of $Z(G)$.*

Corollary 2.8. *Let G be a group having a 3-abelian partition and let $Z(G)$ be its center of order m. The following conditions hold:*

(a) *$G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\kappa(G) = 2^{5m-5}m^{4m-2}$.*

(b) *$G/Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\kappa(G) = 3^{10m-6}m^{9m-2}$.*

(c) *$G/Z(G) \cong \mathbb{S}_3$ and $\kappa(G) = 2^{4m-4}3^{3m-2}m^{6m-1}$.*

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Relation between the solvability of finite groups and their irreducible character degrees

Farideh Shafiei

Abstract
Let G be a finite group and the irreducible character degree set of G is contained in $\{1, a, b, c, ab, ac\}$, where a, b, and c are distinct integers greater than 1. In this paper, we study some structural properties of G reflected by its irreducible character degrees. In particular, we investigate the solvability of G.

Keywords and phrases: character degrees, solvable groups, non-solvable groups, simple groups, Almost simple groups.

2010 *Mathematics subject classification:* Primary: 20C15; Secondary: 20D05, 20D10.

1. Introduction

Throughout this paper, G will be a finite group. We write $\mathrm{dl}(G)$ for the derived length of G when G is solvable and $\mathrm{Irr}(G)$ for the set of irreducible characters of G. The set of character degrees of G is denoted by $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$.

In [3], M. Lewis considered groups G with $\mathrm{cd}(G) = \{1, p, q, r, pq, pr\}$ where p, q, and r are distinct primes. He showed that all such groups are solvable and G is a direct product $A \times B$ where $\mathrm{cd}(A) = \{1, p\}$ and $\mathrm{cd}(B) = \{1, q, r\}$. In [6], the authors proved that if the character degree set of a finite group G is the set $\mathrm{cd}(G) = \{1, m, n, mn\}$ for relatively prime integers m and n, then G is solvable. In [1], K. Aziziheris studied groups whose character degree sets are subsets of the set $\{1, a, b, c, ab, ac\}$ where a, b, and c are pairwise relatively prime integers greater than 1. In this paper, we drop the pairwise relatively primeness hypothesis on a, b, and c.

2. Main Results

We first recall results regarding some character degrees of simple groups of Lie type.

Lemma 2.1. ([[2], Lemma 6.4.7]) *Let S be a simple group of Lie type over a field with characteristic p and St denote the Steinberg character for S. Then* St *is extendible to* $\mathrm{Aut}(S)$ *and its degree* $\mathrm{St}(1) = |S|_p$.

In the following Lemma, we prove that every finite non-abelian simple group of Lie type of rank 1 has two irreducible character which are both extendible to the group of automorphisms of the simple group and these two character have coprime degrees. Hence, for every finite non-abelian simple group S, there exists two irreducible characters which are extendible to $\mathrm{Aut}(S)$. Furthermore, If S is not a simple group of Lie type of rank$\geq$ 2, then the degrees of theses irreducible characters are relatively prime integers.

Lemma 2.2. ([[7], Lemma 4.2]) *Let $S \cong \mathrm{PSL}_2(q)$, where $q = p^f$, and let A be the automorphism group of S. Then there exist nonlinear characters $\eta, \theta \in \mathrm{Irr}(S)$ so that $(\eta(1), \theta(1)) = 1$ and both η and θ extend to A.*

Definition 2.3. *Suppose that S is a finite nonabelian simple group. A group G is said an almost simple group if $S \leq G \leq \mathrm{Aut}(S)$.*

Before stating our main result, we reduce our investigation to almost simple groups. In fact, we show via Theorem 2.4, that no almost simple group with at least five character degrees can occur as our desired groups. This opens up a way to prove the solvability of many groups whose degree sets are contained in the set $\{1, a, b, c, ab, ac\}$, where a, b, and c are integers greater than 1. (see Theorem 2.5.)

Theorem 2.4. ([[7], Theorem 3.1]) *Let G be an almost simple group with $|\mathrm{cd}(G)| \geq 5$. Then* $\mathrm{cd}(G)$ *is not contained in* $\{1, a, b, c, ab, ac\}$*, where a, b, and c are relatively prime integers greater than 1.*

We use the classification of finite simple groups and results of [4] for obtaining our desired consequence in Theorem 2.4.

Using outcomes of [5] for nonsolvable group G with $|\mathrm{cd}(G)| = 4$ and Theorem 2.4, we prove the following theorem which is the main purpose of this paper.

Theorem 2.5. ([[7], Theorem 1.1]) *Let G be a finite group and let a, b, and c be distinct integers greater than 1. If* $\mathrm{cd}(G) \subseteq \{1, a, b, c, ab, ac\}$*, then one of the following statements is true:*

1. *G is solvable;*
2. $\mathrm{cd}(G) = \{1, a, b, c\} = \{1, 9, 10, 16\}$*;*
3. $\mathrm{cd}(G) = \{1, a, b, c\} = \{1, q-1, q, q+1\}$ *for some prime power* $q > 3$.

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Characterization of some almost simple unitary groups by their complex group algebras

Farrokh Shirjian* and Ali Iranmanesh

Abstract

It has been recently proved by Tong-Viet [*J. Algebra* **357** (2012) pp. 61-68] that non-abelian simple groups are uniquely determined by the structure of their complex group algebras. In this paper, we survey the recent improvements of Tong-Viet's result, including our recent work on extending this result to some almost simple groups of Lie type. In particular, we show that some almost simple unitary groups are uniquely determined by the structure of their complex group algebras.

Keywords and phrases: almost simple unitary groups, character degrees, complex group algebras.

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1. Introduction

All groups considered are finite and all characters are complex characters. Let G be a group and $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$ the set of all irreducible characters of G. Also, let $\mathbb{C}G$ denote the complex group algebra of G over the field of complex numbers. The Wedderburn's theorem then yields

$$\mathbb{C}G \cong \bigoplus_{i=1}^{k} M_{\chi_i(1)}(\mathbb{C}).$$

Therefore, the study of complex group algebras and the relations to their base groups plays an important role in group representation theory. A fundamental question in representation theory of finite groups is the extent to which complex group algebra of a finite group determines the group or some of its properties. For instance, it was shown by Isaacs [2] that if $\mathbb{C}G \cong \mathbb{C}H$

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and G has a normal p-complement subgroup N for a prime p, then H has also a normal p-complement M such that $\mathbb{C}N \cong \mathbb{C}M$ and $\mathbb{C}(G/N) \cong \mathbb{C}(H/M)$. In particular, this implies that the nilpotency of a group is preserved by its complex group algebra. It is still unknown whether the solvability of a group is determined by the complex group algebra of the group.

Definition 1.1. *A group G is said to be uniquely determined by the structure of its complex group algebra if for any group H, the isomorphism $\mathbb{C}H \cong \mathbb{C}G$ of complex group algebras implies that $H \cong G$.*

While some properties of a solvable group G might be determined by $\mathbb{C}G$, it is well known that in general, a solvable group is not uniquely determined by the structure of its complex group algebra. For instance, if G is any abelian group of order n, then $\mathbb{C}G$ is isomorphic to a direct sum of n copies of $\mathbb{C}$ so that the complex group algebras of any two abelian groups having the same order are isomorphic. Also, the dihedral group D_8 and the quaternion group Q_8 have identical complex group algebras. In contrast to solvable groups, the non-abelian simple groups have been proved to have a stronger relation to their complex group algebras. Indeed, Tong-Viet proved in [6] that non-abelian simple groups are uniquely determined by the structure of their complex group algebras. He also posed the following question:

Question. Which groups can be uniquely determined by the structure of their complex group algebras?

Definition 1.2. *A group G is called quasi-simple if G is a perfect central extension of a non-abelian simple group; that is, G is a perfect group such that $G/Z(G) \cong S$ for some non-abelian simple group S.*

It has been proved recently that quasi-simple groups are also determined uniquely by the structure of their complex group algebras, see [1, Theorem B].

Definition 1.3. *A group G is called almost simple if there exists a non-abelian simple group S such that $S \leq G \leq Aut(S)$. In this case, the simple group S is called socle of G.*

One of the natural groups to be considered next are almost simple groups. The almost simple groups with an alternating socle have been proved to be uniquely determined by their complex group algebras, see [5]. In this paper, we address the above question for some families of almost simple groups of Lie type.

2. Main Results

In this section, we will state our main results concerning Tong-Viet's question. Let $X_1(G)$ denote the first column of the ordinary character table of G. Note that it means $X_1(G)$ is the multiset of all irreducible character degrees of G counting multiplicities.

Theorem 2.1. *[3, Theorem 1.1] Let q be a prime power and let G be a group such that $X_1(G) = X_1(\mathrm{PGU}_3(q^2))$. Then $G \cong \mathrm{PGU}_3(q^2)$.*

It is well known that $X_1(G)$ is equivalent to the structure of $\mathbb{C}G$. Therefore, according to Theorem 2.1, we have the following corollary:

Corollary 2.2. *[3, Corollary 1.2] Let q be a prime power and let G be a group such that $\mathbb{C}G = \mathbb{C}\mathrm{PGU}_3(q^2)$. Then $G \cong \mathrm{PGU}_3(q^2)$.*

For almost simple groups of Lie type, most of the earlier results are limited only to those groups of small ranks. In the following theorem, we investigate the question above for some almost simple unitary groups of arbitrary rank.

Theorem 2.3. *[4, Main Theorem] Let $n \geq 4$ and q be a prime power. Suppose that G is a finite group such that $\mathrm{PSU}_n(q^2) \leq G \leq \mathrm{PGU}_n(q^2)$, where $q+1$ divides neither of n and $n-1$. Then for any group H with $X_1(H) = X_1(G)$, we have $H \cong G$.*

Using the equivalence of $X_1(H)$ and structure of $\mathbb{C}H$ again, Theorem 2.3 yields:

Corollary 2.4. *[4, Corollary] Let $n \geq 4$ and q be a prime power. Suppose that G is a finite group such that $\mathrm{PSU}_n(q^2) \leq G \leq \mathrm{PGU}_n(q^2)$, where $q+1$ divides neither of n and $n-1$. Then for any group H with $\mathbb{C}H = \mathbb{C}G$, we have $H \cong G$.*

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$\mathrm{IA_z}$-automorphisms of groups

Hamid Taheri* and Mohammad Amin Rostamyari

Abstract

Let G be a group and α be an automorphism of G. Then α is called IA-automorphism if it induces the identity map on the abelianized group G. Also the group of those automorphisms, which fix the centre element-wise, is denoted by $\mathrm{IA}_z(G)$.

In this talk, we prove that the derived subgroups of finite p-groups whose $\mathrm{IA_z}$-automorphisms are the same as central automorphisms, are either cyclic or elementary abelian.

Keywords and phrases: Automorphisms; IA_z-automorphism; Central automorphism .

2010 *Mathematics subject classification:* Primary: 20F28, 20F18; Secondary: 20E10, 20F14.

1. Introduction

In this talk, we study the relationship between $\mathrm{IA_z}$-automorphisms, inner automorphisms and central automorphisms.

An automorphism α of a group G is called IA-*automorphism* if $x^{-1}\alpha(x) \in G'$, for all $x \in G$. This concept was introduced by Bachmuch [1] in 1965, and Gupta [3] in 1981.

We remark that the letters I and A as to remind the reader that are those automorphisms, which induce identity on the abelianized group, G/G'. Also, if $x^{-1}\alpha(x) \in Z(G)$ for all $x \in G$, then we say that α is a *central automorphism*, and if α preserves all conjugacy classes of G, then it is called a *class preserving automorphism*. The set of all such automorphisms are denoted by $\mathrm{IA}(G)$, $\mathrm{Aut_z}(G)$ and $\mathrm{Aut_c}(G)$, respectively. Note that some authors use the notation $\mathrm{Aut_c}(G)$ instead of $\mathrm{Aut_z}(G)$. These concepts were introduced and studied by Curran [2] in 2001 and

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Yadav [14, 15] in 2009 and 2013. The set of all IA-automorphisms, which fix the centre element-wise forms a normal subgroup of $\mathrm{IA}(G)$ and is denoted by $\mathrm{IA_z}(G)$ (see [10, 11] for more information).

For any element x of a group G and automorphism α of $\mathrm{Aut}(G)$, the *autocommutator* of x and α is defined as follows:

$$[x,\alpha] = x^{-1}x^{\alpha} = x^{-1}\alpha(x).$$

Now, using the above notation we have the following

$$\mathrm{Aut_z}(G) = \{\alpha \in \mathrm{Aut}(G) \mid [x,\alpha] \in Z(G), \quad \forall x \in G\},$$

$$\mathrm{Aut_c}(G) = \{\alpha \in \mathrm{Aut}(G) \mid x^{\alpha} \in x^{G}, \quad \forall x \in G\},$$

$$\mathrm{IA_z}(G) = \{\alpha \in \mathrm{Aut}(G) \mid [x,\alpha] \in G',\ \alpha(z) = z, \quad \forall x \in G \textit{ and } z \in Z(G)\}.$$

One can easily check that any class preserving automorphism is an IA-automorphism, which fixes the centre element-wise and hence

$$\mathrm{Inn}(G) \leq \mathrm{Aut_c}(G) \leq \mathrm{IA_z}(G) \leq \mathrm{IA}(G) \leq \mathrm{Aut}(G),$$

$$Z(\mathrm{Inn}(G)) \leq \mathrm{Aut_z}(G) \leq \mathrm{Aut}(G).$$

The following example shows that every $\mathrm{IA_z}$-automorphism is not necessarily inner automorphism.

Example 1.1. *Consider the group*

$$G = \langle a, b, x \mid [a,x] = [b,x] = 1, [a,b] = x^{s},\ s \neq 1\rangle.$$

Clearly, $G' = \langle x^{s}\rangle$, $Z(G) = \langle x\rangle$ *and* $G/Z(G) = \langle \bar{a}, \bar{b}\rangle \cong \mathrm{Inn}(G)$. *The* $\mathrm{IA_z}$*-automorphism* α *defined by* $\alpha(a) = ax^{s}$, $\alpha(b) = bx^{s}$, $\alpha(x) = x$ *is a non-inner automorphism.*

Note that, $\mathrm{Aut_z}(G)$ fixes the derived subgroup G' element-wise. So by using this property we have the following

Lemma 1.2. *For any group G, the central automorphisms commute with* $\mathrm{IA_z}$*-automorphisms of G.*

2. Main Results

Sury [13] generalized the Schur's Theorem as follows:

If G' is finite and $G/Z(G)$ is generated by d elements, then $|\mathrm{Inn}(G)| \leqslant |K(G)|^{d}$, where $K(G) = \langle [x,\alpha] \mid \forall x \in G, \textit{ and } \forall \alpha \in \mathrm{Aut}(G)\rangle$ is the *autocommutator subgroup* of the group G.

Here, we give a further generalized version of the above result, which improves [9].

Theorem 2.1. *Let G be any group with finite derived subgroup. If d is the minimal number of generators of the central factor group of G, then $|\mathrm{IA}_z(G)| \leqslant |G'|^d$.*

Remark 2.2. *One notes that the above theorem improves the result in [9], as* $\mathrm{Inn}(G) \leq \mathrm{Aut}_c(G) \leq \mathrm{IA}_z(G)$.

Clearly, $\mathrm{Aut}_z(G) \cap \mathrm{Inn}(G) = Z(\mathrm{Inn}(G))$, for any group G. Now, we use this property to obtain the following

Lemma 2.3. *Let G be a group such that $Z(G) \leq G'$. Then* $\mathrm{Aut}_z(G) = \mathrm{IA}_z(G)$ *if and only if* $\mathrm{Inn}(G) = Z(\mathrm{Inn}(G))$.

The next result shows that the subgroup $Z(\mathrm{IA}_z(G))$ is between $Z(\mathrm{Inn}(G))$ and $\mathrm{Aut}_z(G)$.

Proposition 2.4. *For any group G,*

$$\mathrm{Aut}_z(G) \cap \mathrm{IA}_z(G) = Z(\mathrm{IA}_z(G)).$$

In 1940, P. Hall [4] introduced the concept of isoclinism between two groups and it was extended to n-isoclinism, which is an equivalent relation among all groups and it is weaker than isomorphism. In 1976, Leedham-Green and McKay [7] extended this concept to isologism with respect to a given variety of groups. There have been extensive studies in this area of mathematics (see [5, 6] for more information).

Definition 2.5. *Let G and H be arbitrary groups and assume $\alpha : G/Z(G) \to H/Z(H)$ and $\beta : G' \to H'$ be isomorphisms such that the following diagram is commutative*

$$\begin{array}{ccc} \dfrac{G}{Z(G)} \times \dfrac{G}{Z(G)} & \stackrel{\alpha\times\alpha}{\longrightarrow} & \dfrac{H}{Z(H)} \times \dfrac{H}{Z(H)} \\ f_1 \downarrow & & \downarrow f_2 \\ G' & \stackrel{\beta}{\longrightarrow} & H', \end{array}$$

where $f_1(g_1Z(G), g_2Z(G)) = [g_1, g_2]$ and $f_2(h_1Z(H), h_2Z(H)) = [h_1, h_2]$, for each $h_i \in \alpha(g_iZ(G))$, $i = 1, 2$, and β is an isomorphism induced by α. Then $(\alpha \times \alpha, \beta)$ is said to be isoclinism, so that G and H are called isoclinic and denoted by $G \sim H$.

Finally, we characterize all finite p-groups, in which their IA_z-automorphisms are equal to central automorphisms. Yadav [15] proved that if two finite groups G and H are isoclinic, then $\mathrm{Aut}_c(G) \cong \mathrm{Aut}_c(H)$. With the same assumption, Pradeep [10] showed that $\mathrm{IA}_z(G) \cong \mathrm{IA}_z(H)$ and he applied his result to prove the following.

Theorem 2.6. ([*10*], Theorem B(1)). *Let G be a finite p-group. Then* $\mathrm{Aut}_z(G) = \mathrm{IA}_z(G)$ *if and only if $Z(G) = G'$.*

Now, using the above theorem, the following results can be deduced.

Lemma 2.7. *Let G be a d-generating finite p-group and $\mathrm{Aut_z}(G) = \mathrm{IA_z}(G)$. Then* $|\mathrm{IA_z}(G)| = |G'|^d$.

Proposition 2.8. *Let finite p-groups G and H be isoclinic and* $\mathrm{Aut_z}(G) = \mathrm{IA_z}(G)$. *Then* $\mathrm{Aut_z}(H) = \mathrm{IA_z}(H)$ *if and only if* $|G| = |H|$.

The following example gives a class of isoclinic groups, whose $\mathrm{IA_z}$-automorphisms are equal with central automorphisms.

Example 2.9. *For any $r, s, t \geqslant 1$ and $1 \leqslant i \leqslant r$. Consider the group*

$$G_i = \langle a, a_1, a_2, \cdots, a_{2s} \mid a^{p^{r+t}} = 1, a_1^{p^r} = \cdots = a_{2s}^{p^r} = a^{p^i}, [a_1, a_2]$$

$$= [a_2, a_3] = \cdots = [a_{2s-1}, a_{2s}] = a^{p^t} \rangle.$$

Clearly, the group G_i is a nilpotent group of class 2 and of order $p^{r(2s+1)+t}$. Also one can easily see that,

$$Z(G_i) \cong C_{p^{r+t}}, \qquad G_i' \cong C_{p^r}, \qquad \frac{G_i}{Z(G_i)} \cong (C_{p^r})^{2s}.$$

Since all G_i''s are cyclic, Proposition 3.2 of [12] and Lemma 3.3 imply that

$$\mathrm{Aut_z}(G_i) = \mathrm{IA_z}(G_i) = Z(\mathrm{Inn}(G_i)) = \mathrm{Inn}(G_i) \cong (C_{p^r})^{2s}.$$

The following lemma of Morigi [8] is useful for our futher investigation.

Lemma 2.10. ([8], *Lemma 0.4) Let G be a finite nilpotent group of class 2. Then $exp(G') = exp(G/Z(G))$ and in the decomposition of $G/Z(G)$ into direct product of cyclic groups at least two factors of maximal orders must occur.*

We remind that the centre of any non-abelian p-group of order p^n lies between p^2 and p^{n-2}.

Theorem 2.11. *Let G be a non-abelian p-group with $\mathrm{Aut_z}(G) = \mathrm{IA_z}(G)$. If $p^3 \leq |G| \leq p^7$, then the derive subgroups of such groups are either cyclic or elementary abelian p-groups.*

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Combinatorics and the Tarski paradox

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Abstract

The Tarski number of a discrete group is the smallest number of parts of its possible paradoxical decompositions. This number is always equal or grater than 4. The only groups with exactly determined Tarski numbers are groups with Tarski numbers 4, 5 and 6. In this talk the attempt is to obtain the paradoxical decomposition of a group by using a matrix combinatorial property. This way, one can find the Tarski number of a given group by counting the number of special paths of a graph associated to that group.

Keywords and phrases: Tarski number, paradoxical decomposition, configuration.
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1. Introduction

Let G be a discrete group. The configurations of G are defined in terms of finite generating sets and finite partitions of G. If $\mathfrak{g} = (g_1, \dots, g_n)$ is a string of elements of G and $\mathcal{E} = \{E_1, \dots, E_m\}$ is a partition of G, a configuration corresponding to $(\mathfrak{g}, \mathcal{E})$ is an $(n+1)$−tuple $C = (c_0, \dots, c_n)$, where $1 \leq c_i \leq m$ for each i, such that there is x in G with $x \in E_{c_0}$ and $g_i x \in E_{c_i}$ for each $1 \leq i \leq n$. The set of all configurations corresponding to the pair $(\mathfrak{g}, \mathcal{E})$ will be denoted by $Con(\mathfrak{g}, \mathcal{E})$. It is shown that groups with the same set of configurations have some common properties. For example they obey the same semigroup laws and have the same Tarski numbers (see [1] and [4]).

In the case that $\mathfrak{g} = \{g_1, \dots, g_n\}$ is a generating set for G, the configuration $C = (c_0, \dots, c_n)$ may be described as a labelled tree which is a subgraph of the Cayley graph of the finitely generated group G and configuration set $Con(\mathfrak{g}, \mathcal{E})$ is a set of rooted trees having height 1.

If $(\mathfrak{g}, \mathcal{E})$ is as above and for each $C \in Con(\mathfrak{g}, \mathcal{E})$

$$x_0(C) = E_{c_0} \cap (\cap_{j=1}^{n} g_j^{-1} E_{c_j}) \quad \text{and} \quad x_j(C) = g_j x_0(C),$$

then it is seen that for any $0 \leq j \leq n$, $\{x_j(C); \quad C \in Con(\mathfrak{g}, \mathcal{E})\}$ is a partition for G. Let $C \in Con(\mathfrak{g}, \mathcal{E})$ and $f \in \ell^1(G)$. Define $f_C = \sum_{x \in x_0(C)} f(x)$. Then we have (see [3])

$$\langle f -_{g_j} f, \chi_{E_i} \rangle = 0, \qquad (1 \leq j \leq n,\ 1 \leq i \leq m)$$

if and only if

$$\sum_{x_0(C)\subseteq E_i} f_C = \sum_{x_j(C)\subseteq E_i} f_C \qquad (1 \le j \le n,\ 1 \le i \le m).$$

For each pair $(\mathfrak{g}, \mathcal{E})$ for G, the system of equations

$$\sum_{x_j(C)\subseteq E_i} f_C = \sum_{x_k(C)\subseteq E_i} f_C, \qquad (1 \le i \le m,\ 0 \le j,k \le n)$$

with variables $f_C,\ \ C \in Con(\mathfrak{g}, \mathcal{E})$ is called the system of configuration equations corresponding to $(\mathfrak{g}, \mathcal{E})$ and is denoted by $Eq(\mathfrak{g}, \mathcal{E})$. By a normalized solution to this system, we mean a solution $(f_C)_C$ such that for each C, $f_C \ge 0$ and $\sum_C f_C = 1$.

2. Main Results

Let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation for the set $\{1, \ldots, n\}$. Then

$$P_\pi = \begin{pmatrix} e_{\pi(1)} \\ e_{\pi(2)} \\ \vdots \\ e_{\pi(n)} \end{pmatrix}$$

is called the permutation matrix associated to π, where e_i denotes the row vector of length n with 1 in the i-th position and 0 otherwise. When the permutation matrix P_π is multiplied with a matrix M from left, $P_\pi M$ will permute the rows of M by π.

If $P = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix}$ is a permutation matrix, by P^+ we mean the matrix with shifted rows, i.e.

$$P^+ = P_\rho P = \begin{pmatrix} P_2 \\ P_3 \\ \vdots \\ P_n \\ P_1 \end{pmatrix},$$

in which ρ is the cyclic permutation $(1\ 2\ \ldots\ n)$. Throughout we use the notation

$$T = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ \vdots & & & & \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix} = \sum_{1\le j\le i\le n} E_{ij},$$

where E_{ij} is the matrix with 1 in ij position and 0 otherwise. When the matrix T is multiplied with a matrix M from left, j-th row of TM will be the sum of j first rows of M.

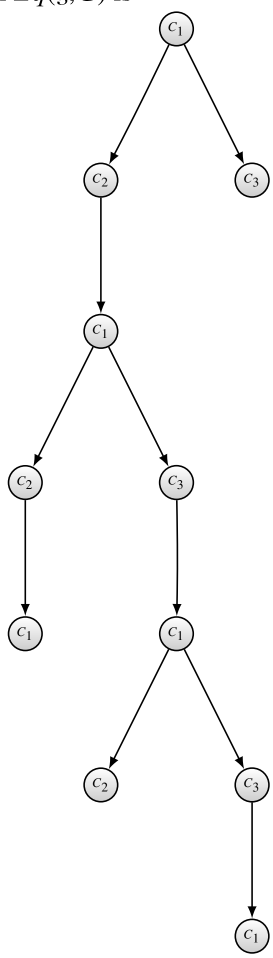
Definition 2.1. Let $\ell \in \mathbb{N}$ and $\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}$ and $\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix}$ be two $(0,1)$-matrices with rows A_i, B_i. Let also the vector $\sum_{i=1}^{n}(B_i - A_i)$ has strictly positive entries. If there exists a permutation matrix P_π such that the matrix

$$TP_\pi(B-A) - P_\pi^{+}A \tag{1}$$

has integer entries equal or greater than -1, we say that the homogenous system of equations $(B-A)X = 0$ is *normal.*

Theorem 2.2. *If a subsystem of $Eq(\mathfrak{g}, \mathcal{E})$ is normal, then G is non-amenable and a paradoxical decomposition of G can be written in terms of $\mathfrak{g}$ and $\mathcal{E}$.*

Example 2.3. Let $Con(\mathfrak{g}, \mathcal{E}) = \{(1,2,2,2), (2,1,2,1), (2,2,1,1)\}$. Then $\tau(G) \leq 5$.
A minimal diagram associated with $Eq(\mathfrak{g}, \mathcal{E})$ is



The number of paths from the top to the bottom is 4. These paths are $[C_1, C_3]$,$[C_1, C_2, C_1, C_2, C_1]$, $[C_1, C_2, C_1, C_3, C_1, C_2]$ and $[C_1, C_2, C_1, C_3, C_1, C_3, C_1]$. Therefore $\tau(G) \leq 4 + 1 = 5$. Indeed we have

$$\begin{aligned}x_0(C_1) =& g_3x_0(C_2) \cup g_3x_0(C_3)\\ x_0(C_2) =& g_1^{-1}x_0(C_1)\\ x_0(C_3) =& g_2x_0(C_1).\end{aligned}$$

Define

$$\begin{aligned}\Delta_1 =& g_3g_1^{-1}g_3g_1^{-1}g_3x_0(C_2),\\ \Delta_2 =& g_3g_1^{-1}g_3g_1^{-1}g_3g_2x_0(C_1,)\\ \Delta_3 =& g_3g_1^{-1}g_3g_2x_0(C_1),\\ \Delta_4 =& g_3x_0(C_3),\\ \Delta_5 =& x_0(C_2) \cup x_0(C_3).\end{aligned}$$

Then $x_0(C_1) = \bigsqcup_{i=1}^4 \Delta_i$. So, $G = \bigsqcup_{i=1}^5 \Delta_i$ and finally we have

$$\begin{aligned}G =& g_2^{-1}g_3^{-1}g_1g_3^{-1}\Delta_3 \bigsqcup g_3^{-1}g_1g_3^{-1}g_1g_3^{-1}\Delta_1 \bigsqcup g_3^{-1}\Delta_4\\ G =& g_2^{-1}g_3^{-1}g_1g_3^{-1}g_1g_3^{-1}\Delta_2 \bigsqcup e\Delta_5,\end{aligned}$$

where e is the identity element of G. We have a complete paradoxical decomposition with five pieces. So, $\tau(G) \leq 5$.

Definition 2.4. *Let $m, n \in \mathbb{N}$. A set of $(n+1)$-tuples $C = \{(c_0^i, \ldots, c_n^i),\ 1 \leq i \leq \ell\}$ with $1 \leq c_j^i \leq m$ and*

$$\cup_{j=1}^n \{c_j^i\} = \{1, \ldots, m\}, \quad (i = 1, \ldots, \ell)$$

is called a pre-configuration set if there exist a group G, a string $\mathfrak{g}$ of elements of G and a partition $\mathcal{E}$ of G such that $Con(\mathfrak{g}, \mathcal{E}) = C$.

In [2] the authors give examples of groups with Tasrki numbers 5 and 6. Now the question is whether we can construct such groups using configurations. Responding to this question depends on knowing that given well-behaved sets are pre-configuration ones. In particular if the following question is answered affirmatively, we will gain a group with tarski number 5.

Question 1. *Is $C = \{(1,2,3,2),(1,3,1,3),(2,1,2,2),(3,3,1,2),(3,3,2,1)\}$ a pre-configuration set?*

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A new characterization of A_5 by Nse

M. Zarrin

Abstract

In this talk, we prove that a nonabelian simple group G has same-order type $\{r, m, n, k\}$ if and only if $G \cong A_5$.

Keywords and phrases: Element order, Same-order type, Nse, Characterization, Simple group .

2010 *Mathematics subject classification:* Primary: 20D60; Secondary: 20D06.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p, $\pi_e(G)$ the set of element orders of G and s_t be the number of elements of order t in G where $t \in \pi_e(G)$.

Let $nse(G) = \{s_t | t \in \pi_e(G)\}$. In fact $nse(G)$ is the set of sizes of elements with the same order in G (the influence of $nse(G)$ on the structure of groups was studied by some authors (for instance see [5], [8]). On the other hand, if we define an equivalence relation $\sim$ as below:

$$\forall\, g, h \in G \;\; g \sim h \iff |g| = |h|,$$

then the set of sizes of equivalence classes with respect to this relation is (which called the same-order type of G and denoted by $\alpha(G)$) equal to the set of sizes of elements with the same order in G. That is,

$$nse(G) = \alpha(G).$$

We say that a group G is a α_n-group if $|\alpha(G)| = n$ (or $|nse(G)| = n$). Clearly the only α_1-groups are 1 , $\mathbb{Z}_2$. In [8], Shen showed that every α_2-group (α_3-group) is nilpotent (solvable, respectively). Furthermore he gave the structure of these groups and he raised the conjecture that if G is a α_n-group, then $|\pi(G)| \leq n$. Most recently, in [3], the authors gave the partial answer to the Shen's conjecture and showed that his conjecture is true for the class of nilpotent groups.

One of the most important problem in group theory is the classification of groups. In [9], the authors showed that a group G is isomorphic to A_5 if and only if $nse(G) = \alpha(A_5) = \{1, 15, 20, 24\}$. Here we give a new characterization of A_5 and show that A_5 is the only nonabelian simple group with the same-order type $\{r, m, n, k\}$ (note that as $s_e = 1$ we may assume that $r = 1$, where e is the trivial element of G, see also [4]).

Theorem 1.1. *Suppose that G is a nonabelian simple group with same-order type $\{1, m, n, k\}$. Then $G \cong A_5$.*

In fact, we show that A_5 is the only nonabelian simple α_4-group. To prove our main result, we shall obtain a nice property of simple groups. In fact we show that for every nonabelian finite simple group G, there exist two odd prime divisors p and q of the order of G such that $s_p \neq s_q$ (see Lemma 2.7 and also Conjecture 1, below).

2. The proof of the main result

First of all note that by Lemma 3 of [9], we can assume that G is finite. To prove the main theorem, we need the following lemmas. Denote by X_n is the set of all elements of order n in a group G.

Lemma 2.1. *Let G be a finite group and k be a positive integer dividing $|G|$. Then $k \mid f(k)$ and $\phi(k) \mid s_k$, where $f(k) = |\{g \in G : g^k = 1\}|$.*

Remark 2.2. *If $m \mid n$ then $\alpha(C_m) \subseteq \alpha(C_n)$, where C_m and C_n are cyclic groups of order m and n, respectively.*

For any prime power q, we denote by $L_n(q)$ the projective special linear group of degree n over the finite field of size q.

Lemma 2.3. *Let $G = L_2(q)$, where q is a prime power of two or q is a Fermat prime and or a Mersenne prime, then there exist $r, s \in \pi(G) \setminus \{2\}$ such that $s_r \neq s_s$.*

Lemma 2.4. *Let $G = Sz(q)$ be the Suzuki group, then there exist $r, s \in \pi(G) \setminus \{2\}$ such that $s_r \neq s_s$.*

Let p be a prime. A group G is called a $C_{p,p}$-group if and only if $p \in \pi(G)$ and and the centralizers of its elements of order p in G are p-groups.

Lemma 2.5. (*see [10]*) *If G is a finite nonabelian simple $C_{2,2}$-group, then G is isomorphic to one of the following groups.*

(a) A_5, A_6, $L_3(4)$;

(b) $L_2(q)$ *where q is a Fermat prime, a Mersenne prime or a prime power of* 2;

(c) $Sz(q)$, *where q is odd prime power of* 2.

Lemma 2.6. (*[6], Lemma 9*) *If $G \neq A_{10}$ is a finite simple group and $\Gamma(G)$ is connected, then there exist three primes $r, s, t \in \pi(G)$ such that $\{rs, tr, ts\} \cap \pi_e(G) = \emptyset$.*

We obtain a interesting property of simple groups.

Lemma 2.7. *Let G be a nonabelian finite simple group. Then there exist two odd prime divisors p and q of the order of G such that $s_p \neq s_q$.*

Corollary 2.8. *Let G be a nonabelian finite simple group. Then there exist two odd prime divisors p and q of the order of G such that $\{1, s_2, s_p, s_q\} \subseteq \alpha(G) = nse(G)$. In fact, every nonabelian finite simple group is a α_n-group with $n \geq 4$ (note that $s_2 \neq 1$, otherwise the center of G, $Z(G) \neq 1$).*

Lemma 2.9. *Assume that G is a group such that $\alpha(G) = \{1, s_r, s_p, s_q\}$. Then*

(i) If $r = 2$, then $\pi(G) = \{2, p, q\}$;

(ii) If $|G|$ is an odd order, then $\pi(G) = \{p, q, r\}$.

We are now ready to conclude the proof of Theorem 1.1.

Proof of Theorem1.1. Since G is simple so $s_2 > 1$, let $s_2 = m$. On the other hand, since G is simple, so $|\pi(G)| \geqslant 3$. By Lemma 2.7, we can choose $\{2, p, q\} \subseteq \pi(G)$, such that $s_2 \neq s_p \neq s_q$ and so by lemma 2.9, $\pi(G) = \{2, p, q\}$. Now it is well-known that the nonabelian simple groups of order divisible by exact three primes are the following eight groups: $L_2(q)$, where $q \in \{5, 7, 8, 9, 17\}$, $L_3(3)$, $U_3(3)$ and $U_4(2)$. Now by the computational group theory system GAP, one can show that all mentioned groups are α_n-group with $5 \leq n$ except A_5.

Finally, in view of Lemma 2.7 and our computation together with the computational group theory system GAP in investigating finite groups of small order suggests that probably in every finite nonabelian simple group the following interesting property is satisfied (in fact, if G is a finite group such that $s_p = s_q$ for some prime divisors p and q of the order of G, then G is not a simple group).

Conjecture 1. *Assume that G be a finite nonabelian simple group. Then for any two prime divisors p and q of the order of G, $s_p \neq s_q$.*

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Some Structural Properties of Power and Commuting Graphs Associated With Finite Groups

Mahsa Zohourattar

Abstract
In this talk we will investigate some properties of the power and commuting graphs associated with finite groups, using their tree-numbers. It has been shown that the simple group $L_2(7)$ can be characterized through the tree-number of its power graph, and the classification of groups with a power-free decomposition is presented. We have also obtained an explicit formula concerning the tree-number of commuting graphs associated with the Suzuki simple groups.

Keywords and phrases: Power graph, commuting graph, tree-number, simple group.
2010 *Mathematics subject classification:* 05C25, 20D05, 20D06.

1. Notation and Definitions

A spanning tree for a graph Γ is a subgraph of Γ which is a tree and contains all the vertices of Γ. The *tree-number* (or *complexity*) of a graph Γ, denoted by $\kappa(\Gamma)$, is the number of spanning trees of Γ (0 if Γ is disconnected). The famous Cayley formula shows that the complexity of the complete graph with n vertices is given by n^{n-2} (Cayley's formula).

In this talk, we shall be concerned with some graphs arising from *finite* groups. Two well known graphs associated with groups are commuting and power graphs, as defined more precisely below. Let G be a finite group and X a nonempty subset of G. Then set (a) The *power graph* $\mathcal{P}(G, X)$, has X as its vertexwith two distinct elements of X joined by an edge when one is a *power* of the other.

(b) The *commuting graph* $C(G, X)$, has X as its vertex set with two distinct elements of X joined by an edge when they *commute* in G.

Clearly, power graph $\mathcal{P}(G,X)$ is a subgraph of commuting graph $\mathcal{C}(G,X)$. In the case when $X = G$, we will simply write $\mathcal{C}(G)$ and $\mathcal{P}(G)$ instead of $\mathcal{C}(G,G)$ and $\mathcal{P}(G,G)$, respectively. Power and commuting graphs have been considered in the literature, see for instance [1, 3–5]. In particular, in [4, Lemma 4.1], it is shown that $\mathcal{P}(G) = \mathcal{C}(G)$ if and only if G is a cyclic group of prime power order, or a generalized quaternion 2-group, or a Frobenius group with kernel a cyclic p-group and complement a cyclic q-group, where p and q are distinct primes. Obviously, when $1 \in X$, the power graph $\mathcal{P}(G,X)$ and the commuting graph $\mathcal{C}(G,X)$ are connected, and we can talk about the complexity of these graphs. For the sake of convenience, we put $\kappa_{\mathcal{P}}(G,X) = \kappa(\mathcal{P}(G,X))$, $\kappa_{\mathcal{P}}(G) = \kappa(\mathcal{P}(G))$, $\kappa_{\mathcal{C}}(G,X) = \kappa(\mathcal{C}(G,X))$ and $\kappa_{\mathcal{C}}(G) = \kappa(\mathcal{C}(G))$. Also, instead of $\kappa_{\mathcal{P}}(G,X)$ and $\kappa_{\mathcal{C}}(G,X)$, we simply write $\kappa_{\mathcal{P}}(X)$ and $\kappa_{\mathcal{C}}(X)$, if it does not lead to confusion.

At last we point out that the content of this talk is a summary and collection of a recent paper by X. Y. Chen, A. R. Moghaddamfar and M. Zohourattar [2].

2. Main Results

A group G from a class $\mathcal{F}$ is said to be recognizable in $\mathcal{F}$ by $\kappa_{\mathcal{P}}(G)$ (shortly, $\kappa_{\mathcal{P}}$-recognizable in $\mathcal{F}$) if every group $H \in \mathcal{F}$ with $\kappa_{\mathcal{P}}(H) = \kappa_{\mathcal{P}}(G)$ is isomorphic to G. In other words, G is $\kappa_{\mathcal{P}}$-recognizable in $\mathcal{F}$ if $h_{\mathcal{F}}(G) = 1$, where $h_{\mathcal{F}}(G)$ is the (possibly infinite) number of pairwise non-isomorphic groups $H \in \mathcal{F}$ with $\kappa_{\mathcal{P}}(H) = \kappa_{\mathcal{P}}(G)$. We denote by $\mathcal{S}$ the classes of all finite simple groups.

Theorem 2.1. *The simple group $L_2(7)$ is $\kappa_{\mathcal{P}}$-recognizable in the class $\mathcal{S}$ of all finite simple groups, that is, $h_{\mathcal{S}}(L_2(7)) = 1$.*

2.1. Power-Free Decompositions a complete graph is a graph in which the vertex set is a complete set. A coclique (edgeless graph or independent set) in Γ is a set of pairwise nonadjacent vertices. A graph Γ is an (m,n)-graph if its vertex set can be partitioned into m cliques $C_1,\ldots,C_m$ and n independent sets $I_1,\ldots,I_n$. In this situation,

$$V_\Gamma = C_1 \uplus C_2 \uplus \cdots \uplus C_m \uplus I_1 \uplus I_2 \uplus \cdots \uplus I_n,$$

is called an (m,n)-split partition of Γ. Also, (m,n)-graphs are a natural generalization of split graphs, which are precisely $(1,1)$-graphs.

Accordingly, we are motivated to make the following definition.

Definition 2.2. Let G be a group and $n \geqslant 1$ an integer. We say that G has an *n-power-free decomposition* if it can be partitioned as a disjoint union of a cyclic p-subgroup C of maximal order and n nonempty subsets $B_1, B_2, \ldots, B_n$:

$$G = C \uplus B_1 \uplus B_2 \uplus \cdots \uplus B_n, \tag{1}$$

such that the B_i's are independent sets in $\mathcal{P}(G)$ and $|B_i| > 1$, for each i. If $n = 1$, we simply say $G = C \uplus B_1$ is a *power-free decomposition* of G.

As the following result shows that there are some examples of groups for which there does not exist any n-power-free decomposition.

Proposition 2.3. *Any cyclic group has no n-power-free decomposition.*

Proposition 2.4. *The generalized quaternion group Q_{2^n}, $n \geqslant 3$, has a 2-power-free decomposition.*

A *universal* vertex is a vertex of a graph that is adjacent to all other vertices of the graph.

Corollary 2.5. *Let G be a group, S the set of universal vertices of the power graph $\mathcal{P}(G)$, and $|S| > 1$. Then G has an n-power-free decomposition if and only if G is isomorphic to a generalized quaternion group.*

Theorem 2.6. *The following conditions on a group G are equivalent:*

(a) *G has a power-free decomposition, $G = C \uplus B$, where C is a cyclic p-subgroup of G.*

(b) *One of the following statements holds:*

  (1) *$p = 2$ and G is an elementary abelian 2-group of order $\geqslant 4$.*

  (2) *$p = 2$ and G is the dihedral group D_{2^m} of order 2^m, for some integer $m \geqslant 3$.*

  (3) *$p > 2$ and G is the dihedral group D_{2p^n} (a Frobenius group) of order $2p^n$ with a cyclic kernel of order p^n.*

2.2. Commuting Graphs Here we consider the problem of finding the tree-number of the commuting graphs associated with a family of finite simple groups. In fact, we shall give an explicit formula for $\kappa_C(\mathrm{Sz}(q))$. Let $G = \mathrm{Sz}(q)$, where $q = 2^{2n+1}$. We begin with some well-known facts about the simple group G (see [6, 7]).

(1) Let $r = 2^{n+1}$. Then $|G| = q^2(q-1)(q^2+1) = q^2(q-1)(q-r+1)(q+r+1)$, and $\mu(G) = \{4, q-1, q-r+1, q+r+1\}$. For convenience, we write $\alpha_q = q-r+1$ and $\beta_q = q+r+1$.

(2) Let P be a Sylow 2-subgroup of G. Then P is a 2-group of order q^2 with $\exp(P) = 4$, which is a TI-subgroup, and $|N_G(P)| = q^2(q-1)$.

(3) Let $A \subset G$ be a cyclic subgroup of order $q-1$. Then A is a TI-subgroup and the normalizer $N_G(A)$ is a dihedral group of order $2(q-1)$.

(4) Let $B \subset G$ be a cyclic subgroup of order α_q. Then B is a TI-subgroup and the normalizer $N_G(B)$ has order $4\alpha_q$.

(5) Let $C \subset G$ be a cyclic subgroup of order β_q. Then C is a TI-subgroup and the normalizer $N_G(C)$ has order $4\beta_q$.

We recall that, in general, a subgroup $H \leqslant G$ is a *TI-subgroup* (trivial intersection subgroup) if for every $g \in G$, either $H^g = H$ or $H \cap H^g = \{1\}$.

Theorem 2.7. *Let $q = 2^{2n+1}$, where $n \geqslant 1$ is an integer. Then, we have*

$$\kappa_C(\mathrm{Sz}(q)) = \left(2^{(q-1)^2} q^{(q^2+q-3)}\right)^{q^2+1} (q-1)^{(q-3)a} (\alpha_q)^{(\alpha_q-2)b} (\beta_q)^{(\beta_q-2)c},$$

where $a = q^2(q^2+1)/2$, $b = q^2(q-1)\beta_q/4$ and $c = q^2(q-1)\alpha_q/4$.

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Generalized exponent of groups

A. Abdollahi, B. Daoud*, M. Farrokhi D. G. and Y. Guerboussa

Abstract

A group G satisfies a positive generalized identity of degree n if there exist elements $g_1, \ldots, g_n \in G$ such that $x^{g_1} \cdots x^{g_n} = 1$ for all $x \in G$. The minimum degree of such an identity is called the generalized exponent of G. Among other things, we prove that every finitely generated solvable group satisfying a positive generalized identity of prime degree is a finite p-group. Consequently, we show that every finite group with a positive generalized identity of degree 5 is a 5-group of exponent dividing 25.

Keywords and phrases: Generalized exponent, polynomial identity.

2010 *Mathematics subject classification:* Primary 05C75; Secondary 05C30, 05E18..

1. Introduction

Let G be a group. The polynomials over G in the indeterminates $x_1, \ldots, x_n$ are defined to be the elements of the free product $G[X] = G * F_X$, where F_X is the free group on the set $X = \{x_1, x_2, \ldots, x_n\}$. Hence, a polynomial $\rho = \rho(x_1, \ldots, x_n)$ on G can be viewed as a word

$$\rho = g_1 w_1 g_2 w_2 \ldots g_n w_n$$

where the g_j's are elements of G, and $w_j = w_j(x_1, \ldots, x_n)$ are elements of the free group F_X.

It is worth noting that the notion of a polynomial can be defined over arbitrary algebraic structures, and can be treated in a general theory via universal algebra, so that the usual notion of polynomials over a commutative ring as well as the one defined over a group are just partic-

* speaker

ular instances of the general notion (see [14]).

Let $u : G \to H$ be a group homomorphism. Then the polynomials over G can be assigned values in H as follows : if $(a_1, \ldots, a_n)$ is an n-tuple of elements of H, then the map $x_i \mapsto a_i$, with $i = 1, \ldots, n$, extends to a homomorphism $F_X \to H$. Hence, by the universal property of free products, the two homomorphisms $u : G \to H$ and $F_X \to H$ determine a unique group morphism $\hat{u} : G[X] \to H$. Hence, each polynomial $\rho = \rho(x_1, \ldots, x_n) \in G[X]$ has an image $\hat{u}(\rho)$ in H that might be denoted $\rho(a_1, \ldots, a_n)$ and called the value of ρ on $(a_1, \ldots, a_n)$. Explicitly, $\rho(a_1, \ldots, a_n)$ is obtained by replacing each indeterminate x_i by the value a_i, and each coefficient g_i occurring in the expression of ρ by $u(g_i) \in H$. It follows that every polynomial $\rho(x_1, \ldots, x_n) \in G[X]$ defines a polynomial mapping

$$\begin{array}{ccc} H^n & \longrightarrow & H \\ (a_1, \ldots, a_n) & \longmapsto & \rho(a_1, \ldots, a_n) \end{array}$$

When speaking about values in G of polynomials over G, i.e., for $H = G$, it should be understood that u is taken to be the identity map $G \to G$.

Let us mention briefly some topics in the literature where the notion of polynomials over groups and their associated mappings appeared.

(a) The group G is said to satisfy a *generalized identity* if for some nontrivial polynomial $\rho(x_1, \ldots, x_n) \in G[X]$, we have $\rho(a_1, \ldots, a_n) = 1$, for all $a_1, \ldots, a_n \in G$. The latter notion can be found in the literature under other names such as *mixed identity* [3, 4], *G-identity* [1], *generalized power law* [**?**], etc. A theory of *generalized varieties of groups* has emerged in the work of V. S. Anashin [4] (we refer the reader to the references therein for earlier work). In analogy with the usual concept of varieties of groups, generalized varieties of groups are classes of groups that satisfy some collection of generalized identities. The subject was investigated more recently in [1] (see also the references therein), and it is related to *algebraic geometry over groups* as we will see below.

(b) Assume that $u : G \to H$ is a monomorphism, that is to say H contains a specified copy of G. In analogy with commutative algebra, such an H plays the role of an algebra over a unital commutative ring, with G playing the role of the coefficient ring. For instance, algebraic sets in H^n can be defined as

$$V_H(S) = \{y \in H^n \mid \rho(y) = 1, \text{ for all } \rho \in S\},$$

where $S \subseteq G[x_1, \ldots, x_n]$. We define, conversely, for each subset $Y \subseteq H^n$, the set $I_H(Y)$, which could be called the ideal of Y, by

$$I_H(Y) = \{\rho \in G[X] \mid \rho(y) = 1, \text{ for all } y \in Y\}.$$

These concepts form the subject of *algebraic geometry over groups*. The theory turned out to have deep similarities to classic algebraic geometry, and it is developed mainly in the papers [7, 13, 15].

(c) On the practical level, polynomial mappings over groups play a central role in non-commutative algebraic dynamics; see [5] and [6, Ch. 6-7]. The subject proved to be quite important for applications in computer sciences, coding theory, and cryptography.

Henceforth, we shall focus only on the polynomials over G in one indeterminate x; the corresponding group will be denoted $G[x]$. So, an element $\rho(x) \in G[x]$ has the form

$$\rho(x) = g_1 x^{n_1} g_2 x^{n_2} \cdots x^{n_k} g_k,$$

where $g_i \in G$ and $n_i \in \mathbb{Z}$, for all i. The integer $\sum_{i=1}^{k} n_i$, will be termed the degree of $\rho(x)$ and will be denoted $\deg \rho$. If $n_i \geq 0$, for all i, then we say that $\rho(x)$ is a positive polynomial.

(d) An element $g \in G$ is called *generalized periodic* if $\rho(g) = 1$, for some nontrivial positive polynomial $\rho \in G[x]$. Generalized periodicity arises in the theory of orderable groups, i.e., groups that can be endowed with a total order which is compatible with the group operation (see [11]). In an orderable group, no nontrivial element could be generalized periodic. The converse to the last result was an open question for several years, and was finally answered negatively in [8].

(e) In the classical setting, and under reasonable conditions, groups that satisfy identities of the form $x^n = 1$ are known to have restricted structure. A prototype here is the affirmative solution by Zelmanov of the Restricted Burnside Problem: *a finitely generated residually finite group which satisfies an identity of the form $x^n = 1$ is finite.* It is natural to wonder whether similar restrictions hold when replacing the last ordinary identities by positive generalized identities. First, a simple verification shows that a group that satisfies a positive generalized identity of degree 2 is abelian of exponent 2. In [10], Endimioni obtained the following results for the positive generalized identities of degree 3 and degree 4.

Theorem 1.1 (Endimioni [10]). *Let G be a group of generalized exponent* 3. *Then*

(1) *G is 3-abelian; that is to say $(ab)^3 = a^3b^3$, for all $a, b \in G$.*

(2) *$G^3 \subseteq Z(G)$;*

(3) *the exponent of G divides 9;*

(4) *G is nilpotent of class at most 3.*

Theorem 1.2 (Endimioni [10]). *Let G be a group of generalized exponent 4. Then*

(1) *G^4 is nilpotent of class ≤ 2;*

(2) *G^8 is abelian.*

2. The main results and related problems

Theorem 2.1. *Every finitely generated solvable group satisfying a positive generalized identity of prime degree p is a finite p-group.*

Corollary 2.2. *A solvable group satisfying a positive polynomial identity of prime degree is a p-group.*

Theorem 2.3. *Every finite group satisfying a positive generalized identity of prime degree $p \leq 5$ is a p-group of exponent dividing p^2.*

Conjecture 1. *The exponent of a finite p-group satisfying a positive generalized identity of degree p is bounded above by p^2.*

Problem 1. *Is every finite group satisfying a positive generalized identity of prime degree nilpotent?*

Problem 2. *Are there only finitely many simple groups that satisfy a positive generalized identity of a given degree n, for every n?*

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Fibonacci length of two classes of 2-generator p-groups of Nilpotent class 2

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Abstract

In this paper, we consider two classes of 2-generator p-groups of Nilpotent class 2 as follows:
$G_1 \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$ where $[a,b] = c, [a,c] = [b,c] = 1, |a| = p^{\alpha}, |b| = p^{\beta}, |c| = p^{\gamma}, \alpha,\beta,\gamma \in N, \alpha \geq \beta \geq \gamma$,
$G_2 \cong \langle a\rangle \rtimes \langle b\rangle$ where $[a,b] = a^{p^{\alpha-\gamma}}, |a| = p^{\alpha}, |b| = p^{\beta}, |[a,b]| = p^{\gamma}, \alpha,\beta,\gamma \in N, \alpha = 2\gamma \text{ and } \beta = \gamma$,
and we will find the Fibonacci length of these groups.

Keywords and phrases: period, finite generated group, Fibonacci length.

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1. Introduction

Definition 1.1. *The t–nacci number sequence, $\{F_n^t\}_0^{\infty}$, is defined by,*

$$F_n^t = F_{n-1}^t + F_{n-2}^t + ... + F_{n-t}^t, \quad n \geq 0,$$

and we seed the sequences with $F_0^t = 0,\ F_1^t = 0,\ \cdots,\ F_{t-2}^t = 0,\ F_{t-1}^t = 1$. We use $K_t(m)$ to denoted the minimal lenght of the period of the series $\{F_n^t(mod\ m)\}_{n=0}^{\infty}$, and call it wall number of m with respective to t–nacci number sequence. (see[4])

Definition 1.2. *Let $G = \langle X\rangle$ be a finitely generated group, where $X = \{a_1, a_2, \cdots, a_n\}$. A Fibonacci sequence of $G = \langle X\rangle$ is the sequence $x_i = a_i,\ 1 \leq i \leq n,\ x_{n+i} = \Pi_{j=1}^{n} x_{i+j-1},\ i \geq 1$. We denote the Fibonacci sequence of the group G, which generated by X, by $F(G;X)$, and denote the minimal period of the sequence $F(G;X)$ by $LEN(G;X)$. (see[1–3])*

* speaker

2. The Fibonacci length of G_1

We consider $G_1 \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a,b] = c, [a,c] = [b,c] = 1, |a| = p^{\alpha}, |b| = p^{\beta}, |c| = p^{\gamma}, \alpha,\beta,\gamma \in N, \alpha \geq \beta \geq \gamma$. In this section, we study the Fibonacci sequence of G_1 with respective to $X = \{c,a,b\}$ and find the period of $F(G_1;X)$. For this purpose, we define the sequences $\{g^3_n\}_1^{\infty}$ and $\{T^3_n\}_1^{\infty}$ of numbers as follows:

$$g^3_n = F^3_{n-3} + F^3_{n-2},$$
$$T^3_1 = 1, T^3_2 = T^3_3 = 0,$$
$$T^3_n = T^3_{n-3} + T^3_{n-2} + T^3_{n-1} - (F^3_{n-4}g^3_{n-2} + (F^3_{n-4} + F^3_{n-3})g^3_{n-1}), \quad n \geq 4.$$

Let $F_i = F^3_i, g_i = g^3_i$, and $T_i = T^3_i$. First, we find a standard form of the 3-nacci sequence $x_4, x_5, \ldots$ of $G_1, n \geq 4$.

Lemma 2.1. *Every element of $F(G_1;X)$ may be presented by $x_n = c^{T_n}a^{g_n}b^{F_{n-1}}$, $n \geq 3$.*

Lemma 2.2. *For every $n \geq 4$, we have,*

$$T_n = F_{n-2} - (F_{n-3} + \Sigma_{i=1}^{n-5} F_{n-(i+3)}(F_{i+1}(F_i + F_{i+1}) + (F_{i+1} + F_{i+2})^2).$$

Lemma 2.3. *If $LEN(G_1;X) = t$, then t is the Least integer such that all of the following equations,*

$$\begin{cases} T_t \equiv 1 \pmod{p^{\gamma}}, \\ T_{t+1} \equiv 0 \pmod{p^{\gamma}}, \\ T_{t+2} \equiv 0 \pmod{p^{\gamma}}, \\ g_t \equiv 0 \pmod{p^{\alpha}}, \\ g_{t+1} \equiv 1 \pmod{p^{\alpha}}, \\ g_{t+2} \equiv 0 \pmod{p^{\alpha}}, \\ F_t \equiv 0 \pmod{p^{\beta}}, \\ F_{t+1} \equiv 0 \pmod{p^{\beta}}, \\ F_{t+2} \equiv 1 \pmod{p^{\beta}}. \end{cases}$$

hold. Moreover, $K_3(m)$ divides $LEN(G_1;X)$.

Example 2.4. *(i) For integer $\alpha = \beta = \gamma = 1$ and $p = 5$, we have $K_3(5) = 31$, $LEN(G_1;X) = 31$. Since, $x_1 = c, x_2 = a, x_3 = b, x_4 = cab, x_5 = a^2b^2, x_6 = c^{-4}a^3b^4, \ldots, x_{32} = x_{31+1} = c^{-35901699729984}a^{24548655}b^{29249425} = c, x_{33} = x_{31+2} = a, x_{34} = x_{31+3} = b, \ldots$. Consequently, $x_{32} = x_{31+1} = x_1, x_{33} = x_{31+2} = x_2, x_{34} = x_{33+3} = x_3$. Therefore, $LEN(G_1;X) = K_3(5)$.*
(ii) For integer $\alpha = 2, \beta = \gamma = 1$ and $p = 7$, we have $K_3(7^2) = 336$, $LEN(G_1;X) = 336$. Since, $x_1 = c, x_2 = a, x_3 = b, x_4 = cab, x_5 = a^2b^2, x_6 = c^{-4}a^3b^4, \ldots, x_{49} = c^0a^{14}b^0 \ldots, x_{50} =$

$c^0a^{29}b^0, \ldots x_{337} = x_{336+1} = c, x_{338} = x_{338+2} = a, x_{339} = x_{339+3} = b, \ldots$.
Consequently, $x_{337} = x_{336+1} = x_1, x_{338} = x_{336+2} = x_2, x_{339} = x_{336+3} = x_3$. *Therefore,* $LEN(G_1; X) = K_3(7^2)$.

The following theorem is the main results of this section.

Theorem 2.5. *For integers $\alpha \geq \beta \geq \gamma$, we have, $LEN(G_1; X) = K_3(p^\alpha)$.*

Corollary 2.6. *We consider G_1 with respective to $X = \{a, b, c\}$. Then every element of $F(G_1; X)$ may be presented by $x_n = a^{F_{n-2}}b^{u_n}c^{d_n}$, where*

$$u_n = F_{n-2} + F_{n-3}, \quad d_n = d_{n-1} + d_{n-2} + d_{n-3} - (F_{n-4}u_{n-2} + (u_{n-2} + u_{n-3})F_{n-3}),$$
$$d_1 = d_2 = 0, \quad d_3 = 1, \ n \geq 4.$$

We get the minimal period of the sequence $F(G_1; X)$ is $K_3(p^\alpha)$.

3. The Fibonacci length of G_2

Here, we discuss the period of the generalized order 2–nacci sequence of $G_2 \cong \langle a \rangle \rtimes \langle b \rangle$ where $[a, b] = a^{p^{\alpha-\gamma}}, |a| = p^\alpha, |b| = p^\beta, |[a, b]| = p^\gamma, \alpha, \beta, \gamma \in N, \alpha = 2\gamma$ and $\beta = \gamma$. First, We find a standard form of Fibonacci sequence $x_4, x_5, \ldots$ of G_2, with respective to $X = \{a, b\}$. For this, we need the following sequence:

$$h^2_1 = 1, h^2_2 = 0, \quad h^2_3 = p^\gamma + 1, \quad h^2_n = h^2_{n-2}(p^\gamma + 1)^{F^2_{n-2}} + h^2_{n-1} \ \ n \geq 4.$$

Let $F_i = F^2_i$ and $h_n = h^2_n$.

Lemma 3.1. *Every element of $F_2(G_2; X)$ may be presented by $x_n = b^{F_{n-1}}a^{h_n}$, $n \geq 4$.*

Lemma 3.2. *For every $n \geq 5$, we have,*

$$h_n \equiv F_{n-2} + p^\gamma(F_{n-2} + \Sigma^{n-4}_{i=1}F_iF_{i+1}), \ \ (mod\ (p^{2\gamma})).$$

Example 3.3. *For integer $\alpha = 2, \beta = \gamma = 1$ and $p = 3$. Since we have $K_2(3) = 8$, by relations of G_2, we obtain the Fibonacci length of G_2 as follows:*

$$x_1 = a, \ \ x_2 = b, \ \ x_3 = ba^4, \ldots, \ \ x_9 = b^{21}a^{4694688993604} \neq a,$$
$$x_{10} = b^{34}a^3, \ldots, \ \ x_{25} = a, \ \ x_{26} = b, \ldots.$$

Consequently,

$$x_{25} = x_{24+1} = x_1, \quad x_{26} = x_{24+2} = x_2.$$

Then $LEN(G_2; X) = 24$.

We can now proceed to the main results of this section.

Theorem 3.4. *For integers $\alpha = 2\gamma$ and $\beta = \gamma$, we will have $P_2(G_2; X) = K_2(p^{2\gamma})$.*

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Group homomorphism on ultra-group

PARVANEH ZOLFAGHARI

Abstract

This paper presents an ultra-group ${}_HM$ which depends on the group G and its subgroup H have been defined. Further, it is proving that if ${}_HM$ is an ultra-group over group G and f is a monomorphism from group G to group G', then $f(M)$ is a transversal of subgroup $f(H)$ over the group $f(G)$. Finally, if f is an isomorphism, then $f(M)$ is an ultra-group of subgroup $f(H)$ over the group G'.

Keywords and phrases: Ultra-group, transversal, homomorphism .

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1. Introduction

Let H be a subgroup of the group G. A subset M of G is called a (right) transversal to H in G if $G = \bigcup_{m\in M} Hm$. Therefore the pair (H, M) is a (right) transversal if and only if the subset M of the group G obtained by selecting one and only one member from each right coset of G modulo subgroup H (see [4]). Therefore for any elements $m \in M$ and $h \in H$ there exists unique elements $h' \in H$ and $m' \in M$ such that $mh = h'm'$. We denote h' and m' by ${}^mh$ and m^h, respectively. For any elements $m_1, m_2 \in M$ there exist unique elements $[m_1, m_2] \in M$ and ${}^{(m_1,m_2)}h \in H$ such that $m_1m_2={}^{(m_1,m_2)}h[m_1, m_2]$. Furthermore for every element $a \in M$, there exists $a^{(-1)} \in H$ and $a^{[-1]} \in M$ such that $a^{-1} = a^{(-1)}a^{[-1]}$. Throughout the paper, all the necessary definitions and preliminary statements may be found in (see[1, 2]).

Definition 1.1. *A right transversal set M of subgroup H over group G with a binary operation $\alpha : M\times M \to M$ and unary operation $\beta_h : M \to M$ defined by $\alpha((m_1, m_2)) := [m_1, m_2]$ and $\beta_h(m) := m^h$ for all $h \in H$ is called right ultra-group.*

We use the notation ${}_HM$ (M_H) to represent the right (left) ultra-group M of subgroup H. From now on, unless specified otherwise, ultra-group means right ultra group. A subset $S \subseteq {}_HM$ which contains e, is called a *subultra-group* of H over G, if S is closed under the operations α and β_h in the Definition 1.1. It is obvious that $\{e\}$ is a trivial subultra-group for all ultra-groups ${}_HM$ where e is the identity element of H. Suppose A, B are two subsets of the ultra-group ${}_HM$. Moreover we use the notation $[A, B]$ for the set of all $[a, b]$, where $a \in A$ and $b \in B$. If B is a singleton $\{b\}$, then we denote $[A, B]$ by $[A, b]$. A subultra-group N of ${}_HM$ is called *normal* if $[a, [N, b]] = [N, [a, b]]$, for all $a, b \in_H M$. In addition $[N, S]$ is a subultra-group of ${}_HM$, where S is a subultra-group of ${}_HM$. Moreover, $[N, S]$ is a normal subultra-group of ${}_HM$ if S is also normal subultra-group of ${}_HM$.
Similarly we can define left ultra-groups by use of left unitary complimentary set. A homomorphism is a map that preserves selected structure between two algebraic structures, with the structure to be preserved being given by the naming of the homomorphism.

Definition 1.2. *Suppose ${}_{H_i}M_i$ is ultra-group of H_i over group G_i, $i = 1, 2$ and φ is a group homomorphism between two subgroups H_1 and H_2. A function $f : {}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ is an ultra-group homomorphism provided that for all $m, m_1, m_2 \in {}_{H_1}M_1$ and $h \in H_1$.*
(i) $f([m_1, m_2]) = [f(m_1), f(m_2)]$,
(ii) $(f(m))^{\varphi(h)} = f(m^h)$.

If f is a surjective and injective ultra-group homomorphism, then we call it isomorphism and denote it by ${}_{H_1}M_1 \cong {}_{H_2}M_2$. All homomorphism between the two ultra-groups preserves the identity and left inverse elements. If S is a subultra-group of ${}_{H_1}M_1$ and φ is surjective homomorphism, then $f(S)$ is a subultra-group of ${}_{H_2}M_2$. Moreover, $\mathrm{Ker}(f) = \{m \in_{H_1} M_1 \mid f(m) = e_{{}_{H_2}M_2}\}$ is a normal subultra-group of ${}_{H_1}M_1$, where $f : {}_{H_1}M_1 \to {}_{H_2}M_2$ is an ultra-group homomorphism (see [3, 5]).

2. Main Results

In this article, we show that every isomorphism group is a homomorphism ultra-group.

Theorem 2.1. *Let $f : G \longrightarrow G'$ be a monomorphism between two groups and (H, M) a pair transversal of subgroup H and subset M of the group G. Then $(f(H), f(M))$ is a transversal of subgroup $f(H)$ and subset $f(M)$ of the group $f(G)$.*

Thus, monomorphism between two groups G and G' preserve ultra-groups over groups G and G'.

Theorem 2.2. *If f is an isomorphism between two groups G and G' and (H, M) a pair* *transversal of subgroup H and subset M of the group G, then $f(M)$ is an ultra-group of subgroup $f(H)$ over the group G' and f is an isomorphism between two ultra-groups ${}_HM$ over group G and ${}_{f(H)}f(M)$ over group G'.*

An automorphism is an isomorphism from an object to itself. The set of all automorphisms of an object forms an ultra-group, called the automorphism ultra-group.

Theorem 2.3. *Let H be a subgroup of a group G and $\mathcal{W}$ is the set of all ultra-groups of the subgroup H over group G. Then $Aut(G)$ is a subset of intersection of all $Aut({}_HM)$, where ${}_HM \in \mathcal{W}$.*

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